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THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING
Instrumentation Engineering Program
Department of Aeronautical and Astronautical Engineering

Technical Report

# STATIONARY POINT PROCESSES AND THEIR APPLICATION TO RANDOM SAMPLING OF STOCHASTIC PROCESSES

Oscar Aszer Zelig Leneman



ORA Project 02905



administered through

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

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# INTRODUCTION

The problem of reconstructing a signal x(t) from a set of sampled values $\{x(t_n)\}$ [the $t_n$ denote the sampling instants] is of interest in both communications and control. The type of sampling where $t_{n+1} - t_n = T =$ constant, for all n, is known as periodic sampling and this special case has received considerable attention. However, the case of periodic sampling is idealized and is, in practice, extremely difficult to obtain, as imperfections in the sampling mechanism will always give some uncertainty about the exact location of each sampling instant $t_n$. This uncertainty is usually known as "time-jitter" and is often described as a random error in timing. In other words, the exact location of the sampling instants $t_n$ is uncertain or random, and the presumed periodic sampling becomes a random sampling. It is clear that the time-jitter error will affect the design of the optimum recovery scheme of x(t) and its effects must be studied. Similarly, random sampling occurs in sampled-data systems because of inaccuracies in the equipment which is designed to sample at constant intervals. Random sampling also can occur in a multi-loop system because of a random time delay which must elapse before a digital computer again becomes available to carry out control computations required by a particular feedback loop of the system. Also, in radar problems [with nominal scanning period T ], it happens occasionally that some samples of the return signals are absent or rejected due to excessive noise or other interference; this is referred to as the miss [or skip] problem in radar. Thus the availability of data at the nominal sampling instants may only be described probabilistically.

Furthermore, there also exists the possibility of intentional random sampling. For instance, it may be convenient to have a random scheme of sampling in order to reduce the susceptability of some systems to jamming or similar interference. Also, it could be suggested that, for reasons of economy, a time-shared digital computer, used for the control of a number of plant processes, may be made available to any particular process at random, rather than at specified instants of time. Moreover, it can be shown (ref S.2. ) that random sampling is of practical interest in determining the power spectra $\phi_x(\omega)$ of a weakly stationary random process x(t) from a set of samples $\{x(t_n)\}$ . It is well-known that the determination is unamgibuous if the spectra is band-limited and the sampling is periodic with a frequency greater than the Nyquist frequency. But if the spectra is not band-limited or it is not certain whether it is, then there exists a whole class of power spectra [known as "aliases" of $\phi_x(\omega)$] which are compatible with the periodic sample values. Shapiro and Silverman have shown however that some random sampling schemes e. g. Poisson sampling will eliminate the aliases and hence allow an unambiguous recovery of the spectra $\phi_x(\omega)$.

The effect of random sampling in association to problems of interpolation and of sampled-data systems has been partially investigated, leading to a certain number of results. Kalman (ref K. 1. ) was concerned with the stability of a first-order sampled-data system in which the sampler operates at random times [ and assuming that the time intervals are independent random variables] . This synthesis procedure is based on the technique of dynamic programming. Bergen [ref B. 4, B. 5] has studied the statistics of the output of a linear time-invariant system which is preceded by a sampler switching at random instants of time [with the time intervals being independent random variables]. His synthesis procedure for optimum designs is based on the minimum mean square error criterion. Balakrishnan (ref B.2 . ) investigated the problem of optimum non-realizable interpolation of band-limited signals when time-jitter error is present during periodic sampling. Brown (ref B.9 ) generalized slightly the preceding problem by considering non-band limited signals and two sources of time-jitter errors ["read-in" jitter and "read-out" jitter]. Adomian (ref A. 1. ), using the periodogram method, calculated the spectral density of the output of a zero-order hold preceded by a sampler which operates at random instants of time [the time intervals being independent random variables].

In the present work, among the other things, we have attempted to extend some of the studies of the preceding authors. Using the minimum mean square error criterion, we investigate the problem of optimum linear interpolation [non-realizable solutions] of a signal from samples taken at random instants of time [e.g. nearly-periodic sampling with skips, Poisson sampling, etc....], with errors in amplitude also present The preceding study is quite simply performed with the use of an "improper" random process, the random impulse process defined as $s(t) = \sum_{n=-\infty}^{\infty} \alpha_n \delta(t - t_n)$ where the random sampling instants $t_n$ constitute a so-called stationary point process [section 1.2.] which is independent of the stationary random process $\{\alpha_n\}$. Moreover, some of the statistical characteristics of "secondary processes" y(t) defined as $y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \eta(t, t_n)$, where $\eta(t, u)$ denotes a deterministic function, are considered and studied. As a result, Campbell's theorem is easily obtained. Furthermore, a whole variety of step-wise random processes are introduced and their second-order statistics are investigated either in the time-domain or in the frequency domain. In particular, we consider the statistics of the output of a zero-order hold [in a sampled-data system] preceded by a sampler which switches at random instants of time; the results obtained may be useful for the statistical design of optimum random sampled-data systems.

In order to carry on the preceding studies, it was necessary to define a statistical representation of random points on the line, to introduce the concept of a stationary point process and to study its properties. This concept has been inspired by the work of McFadden [ref M.1., M.2.], who used a somewhat different approach from that used here.

Many examples are given.

# CHAPTER I

## Stationary Point Processes

Summary

This chapter introduces the concept of point processes and defines the class of stationary point processes. Some general properties are investigated and particular attention is devoted to the statistical behavior of the random number of points (or occurrences) which fall in a given time interval. A few methods for generating point processes which are stationary are indicated and various examples are given.

### 1.1 Recurrence patterns and point processes

#### 1.1.1 Definitions

In what follows, we attempt to describe statistically the random instants of occurence of an enumerable sequence of specific events (e.g., electrons emitted in a vacuum tube, or customers entering a store).

The arrival pattern: forward point process.

Suppose we choose a fixed instant t and then, without any knowledge of the sequence of events, inquire about the random time of occurence of the first event after t. We may denote that instant by $t_1$ and by $t_2, t_3 \ldots$ the successive instants of events in sequence where

$$t \leq t_1 \leq t_2 \leq \cdots \leq t_n \tag{1.1.1}$$

for all n. By definition, the set of random instants $\{t_n\}_{n=1,2\ldots}$ will be called a forward point process.

At this point, it is of interest to mention that the ordering of events according to (1.1.1) is not always possible. For example, consider the case of events occurring at all rational instants of time: those events are countable but they can not be ordered according to (1.1.1). Such a situation will be undesirable and, from now on, we shall only consider occurrences satisfying the ordering equation (1.1.1)

Next, we introduce the random intervals

$$L_n(t) = t_n - t \tag{1.1.2}$$

$$x_n(t) = t_{n+1} - t_n \quad , \ n = 1, 2, \ldots \tag{1.1.3}$$

where $L_1(t)$ is known as the first passage time and where $x_n(t)$ is often called the inter-arrival time between consecutive events (ref T. 2). With these notations, eqn. (1.1.1) gives

$$0 \leq L_1(t) \leq L_2(t) \leq \cdots \leq L_n(t) \tag{1.1.4}$$

and it is clear that the arrival pattern of events (or occurrences) is statistically described by the random process $\{L_n(t)\}_{n=1,2,\ldots}$ .

Such a random process is constantly used in the study of queueing and renewal theory (ref F.1, C.1, T.2, P.2). The most frequent case is the one where $L_1(t)$, $x_1(t), \ldots$ $x_n(t)$ are positive, mutually independent random variables such that all the $x_n(t)$ are identically distributed.

The departure pattern: backward point process

In a similar way, in order to describe the departure pattern of this sequence of events, we also introduce the concept of a backward point process. This process is defined by the instants $\{t_{-n}\}_{n=1,2\ldots}$ where $t_{-1}$ denotes the instant of occurrence of the most recent event before the instant t, and where $t_{-2}$, $t_{-3} \ldots t_{-n}$ are the backward successive instants

$$t_{-n} \leq \cdots \leq t_{-2} \leq t_{-1} \leq t \ . \tag{1.1.5}$$

Let

$$L_{-n}(t) = t - t_{-n} \tag{1.1.6}$$

and

$$x_{-n}(t) = t_{-n} - t_{-(n+1)} \, , \quad n = 1, 2, \ldots \ . \tag{1.1.7}$$

Then the departure pattern of events is statistically described by the random process $\{L_{-n}(t)\}_{n=1,2,\ldots}$ .

The complete recurrence pattern: point process

Finally, there are situations when one inquires jointly about backward and forward occurrences. In this case, by using the previous notations, we are led to the point process $\{t_n\}_{n=\pm 1,\pm 2\ldots}$ which is described by the random process $\{L_n(t)\}_{n=\pm 1,\pm 2,\ldots}$ .

Symmetric point process

If the random processes $\{L_n(t)\}_{n=1,2,\ldots}$ and $\{L_{-n}(t)\}_{n=1,2,\ldots}$ are statistically identical, we say that the point process $\{t_n\}_{n=\pm 1,\pm 2,\ldots}$ is symmetric for the instant t.

1.1.2 The number of occurrences in a given time interval

For simplicity, we shall only consider forward occurrences, the extension to backward occurrences being similar. We shall consistently use the abridged notation

$$\{L_n(t) \le \tau\}$$

for the set

$$\{\omega | L_n(t,\omega) \le \tau\} \ , \quad \omega \in \Omega \ ,$$

$\Omega$ being the basic probability space. As usual, the letters P and E will stand respectively for probability and expectation.

We denote by $G_n(\tau)_t$ the distribution function associated with the random variable $L_n(t)$, that is

$$G_n(\tau)_t = P[L_n(t) \le \tau] \ . \tag{1.1.8}$$

Because of (1.1.4), we have

$$G_n(\tau)_t = 0, \quad \tau < 0 \tag{1.1.9}$$

$$G_{n+1}(\tau)_t \leq G_n(\tau)_t \tag{1.1.10}$$

$$n = 1, 2, \ldots$$

Let $N(t + x, \tau)$ be the random number of events which occur in the interval $(t + x,\ t + x + \tau]$, $x \geq 0$, $\tau > 0$, and let

$$p(n, \tau)_{t+x} = P[N(t + x, \tau) = n] . \tag{1.1.11}$$

observing that

$$\{N(t, \tau) = 0\} = \{L_1(t) > \tau\} \tag{1.1.12}$$

$$\{N(t, \tau) = n\} = \{(L_n(t) \leq \tau) \cap (L_{n+1}(t) > \tau)\}$$

$$= \{(L_{n+1}(t) > \tau) - (L_n(t) > \tau)\} \tag{1.1.13}$$

and noticing that

$$\{L_n(t) > \tau\} \subset \{L_{n+1}(t) > \tau\} \tag{1.1.14}$$

we obtain

$$p(0, \tau)_t = 1 - G_1(\tau)_t \tag{1.1.15}$$

$$p(n, \tau)_t = G_n(\tau)_t - G_{n+1}(\tau)_t\,, \quad n \geq 1. \tag{1.1.16}$$

Similarly, we find

$$p(0, \tau)_{t+x} = P[L_1(t) > \tau + x]$$

$$+ \sum_{k=1}^{\infty} P[L_k(t) \leq x) \cap (L_{k+1}(t) > \tau + x)] \tag{1.1.17}$$

$$p(n, \tau)_{t+x} = P[(x < L_1(t) \leq L_n(t) \leq \tau + x) \cap (L_{n+1}(t) > \tau + x)]$$

$$+ \sum_{k=1}^{\infty} P[(L_k(t) \leq x) \cap (x < L_{k+1}(t) \leq L_{n+k}(t) \leq \tau + x) \cap (L_{n+k+1}(t)$$

$$> \tau + x)] \quad , \quad n \geq 1 . \tag{1.1.18}$$

### 1.1.3 Conclusions: Example

A comparison of (1.1.16) with (1.1.18) shows that, in general, the random variables $N(t, \tau)$ and $N(t + x, \tau)$ are not identically distributed. This result is obviously expected in many practical situations: for example, the arrival pattern of customers entering a store may vary considerably at different hours of the day. But, on the other hand, there are many physical situations in which one feels that the probability structure of the recurrence pattern for a certain type of event does not vary with time; for instance, this may be the case when one studies the emission pattern of electrons in a vacuum tube, assuming that the tube is in steady-state operation. Consequently, it is useful to define a concept of stationarity for point processes, and we shall do so in the next section.

Another point of concern is the following one: in many physical situations, one would like to think of $N(t + x, \tau)$ as a finite-valued random variable so that

$$\sum_{n=0}^{\infty} p(n, \tau)_{t+x} = 1. \tag{1.1.19}$$

But this condition need not always hold. For example, consider the forward point process defined as

$$t_n = t + \alpha + 1 - \frac{1}{n}, \quad n = 1, 2, \ldots \infty \tag{1.1.20}$$

where $\alpha$ is a random variable uniformly distributed between zero and one. From

$$L_n(t) = t_n - t = \alpha + 1 - \frac{1}{n}, \; n = 1, 2, \ldots \tag{1.1.21}$$

we obtain

$$G_n(\tau)_t = \begin{cases} 0 & , \quad \tau \le 1 - \frac{1}{n} \\ \tau - (1 - \frac{1}{n}) & , \quad 1 - \frac{1}{n} < \tau \le 2 - \frac{1}{n} \\ 1 & , \quad \tau > 2 - \frac{1}{n} \end{cases} \tag{1.1.22}$$

Equations (1.1.15), (1.1.16) give

$$p(n, 3)_t = 0 \quad , \quad n = 0, 1, 2, \ldots \tag{1.1.23}$$

so that

$$\sum_{n=0}^{\infty} p(n, 3)_t = 0 \tag{1.1.24}$$

thus showing that N(t, 3) is not finite-valued, as could be expected by noting that the interval (t, t + 3] contains a limit point.

## 1.2 Stationary point process

### 1.2.1 Definitions

The point process described by $\{L_n(t)\}_{n=\pm 1, \pm 2, \ldots}$ is called stationary if the joint distribution function of the random variables

$$\{L_{n_1}(t + h),\ L_{n_2}(t + h), \ldots, L_{n_k}(t + h)\}$$

is identical with the joint distribution function of the random variables

$$\{L_{n_1}(t)\ ,\ L_{n_2}(t), \ldots, L_{n_k}(t)\}$$

for all real values of h and all $n_k \in \{\pm 1,\ \pm 2, \ldots\}$.

In a similar way one may define stationarity for the forward point process $\{L_n(t)\}_{n=1,2,\ldots}$ and the backward point process $\{L_{-n}(t)\}_{n=1,2,\ldots}$.

### 1.2.2 Fundamental theorem:

A point process is stationary if, and only if, the associated forward (or backward) point process is stationary.

Proof:

n) The necessary part is obvious.

s) We have to show that the knowledge of the stationary forward process $\{L_n(t)\}_{n=1,2,\ldots}$ determines that of the process $\{L_n(t)\}_{n=\pm 1, \pm 2, \ldots}$.

First, let us determine the distribution function associated with $L_{-n}(t)$.

Letting

$$E_n(t, \tau) = \begin{Bmatrix} \text{at least n occurrences} \\ \text{in the interval } (t, t+\tau] \end{Bmatrix}, \tau > 0 \qquad (1.2.1)$$

and observing that

$$E_n(t-\tau, \tau) = \{L_{-n}(t) \le \tau\} \qquad (1.2.2)$$

$$E_n(t-\tau, \tau) = \{L_n(t-\tau) \le \tau\} \qquad (1.2.3)$$

we obtain

$$P[L_{-n}(t) \le \tau] = P[L_n(t-\tau) \le \tau] \qquad (1.2.4)$$

The forward point process is stationary so that $L_n(t-\tau)$ does not depend on its argument (whenever there is no ambiguity, we shall write $L_n$ instead of $L_n(t-\tau)$). Defining

$$G_n(\tau) = P[L_n \le \tau] \qquad (1.2.5)$$

we obtain

$$P[L_{-n}(t) \le \tau] = G_n(\tau) \qquad (1.2.6)$$

so that the distribution of $L_{-n}(t)$ does not depend on its argument and is the same as that of $L_n$.

Next, we consider higher order distribution functions. It will be sufficient to consider an example. For instance

$$P[(L_{-1}(t) \le \tau_1) \cap (L_{-2}(t) \le \tau_1 + \tau_2)]$$

$$= P[L_{-2}(t) \le \tau_1] + P[L_{-1}(t) \le \tau_1) \cap (\tau_1 < L_{-2}(t) \le \tau_1 + \tau_2)]$$

$$= P[L_2(t - \tau_1) \le \tau_1] + \sum_{n=1}^{\infty} P[(L_n(t - \tau_1 - \tau_2) \le \tau_2) \cap$$

$$(\tau_2 < L_{n+1}(t - \tau_1 - \tau_2) \le \tau_1 + \tau_2) \cap (L_{n+2}(t - \tau_1 - \tau_2) > \tau_1 + \tau_2)]$$

$$= P[L_2 \le \tau_1] + \sum_{n=1}^{\infty} P[(L_n \le \tau_2) \cap (\tau_2 < L_{n+1} \le \tau_1 + \tau_2) \cap (L_{n+2} > \tau_1 + \tau_2)] \qquad (1.2.7)$$

This theorem can be intuitively expected from the following argument: For $t = -\infty$, the entire point process appears as a forward point process and since $\{L_n(t)\}_{n=1,2,\ldots}$ does not depend on t, one may choose $t = -\infty$.

Notice that although $L_n$ and $L_{-n}$ are identically distributed, equation (1. 2. 7) shows that, in general, the processes $\{L_n\}_{n=1,2,\ldots}$ and $\{L_{-n}\}_{n=1,2,\ldots}$ are not statistically identical, so that a stationary point process is not necessarily symmetric.

1. 2. 3 Discussion

We would like to show that, in general, it is hard to determine whether a point process is stationary. Suppose we are given a point process defined by $\{L_n(t)\}_{n=\pm 1,\pm 2,\ldots}$ and we inquire whether this point process is stationary or not. From the preceding theorem, it is sufficient to determine the random process $\{L_n(t+h)\}_{n=1,2,\ldots}$ and verify whether or not the joint distributions of this random process are independent of h, for all real values of h. This verification is usually very difficult. For example, let us determine the distribution function associated with $L_1(t+h)$, where we take $h > 0$. From the probabilistic interpretation of

$$\{L_1(t+h) \leqslant \tau\}$$

we find that

$$P[L_1(t+h) \le \tau] = P[h < L_1(t) \le \tau + h]$$

$$+\sum_{n=1}^{\infty} P[(L_n(t) \le h) \cap (h < L_{n+1}(t) \le \tau + h)] , \qquad (1.\ 2.\ 8)$$

which can be evaluated from the given random process $\{L_n(t)\}_{n=\pm 1,\pm 2,\ldots}$ .
If the point process is to be stationary, we must obtain

$$P[L_1(t+h) \le \tau] = P[L_1(t) \le \tau] .$$

The difficulties increase considerably when we attempt to determine higher order distribution functions.

In spite of the preceding pessimistic comments, the class of stationary point processes is not empty: the most familiar examples include the Poisson renewal point process (ref P. 2, T. 1, . . . ) and the periodic point process (periodic occurrences, with a random starting time uniformly distributed over the period). A few additional examples will be given in section 1. 4.

### 1. 2. 4 Some general properties of a stationary point process

#### 1. 2. 4. 1 First order properties

Equations (1. 2. 5), (1. 1. 10), (1. 2. 1) give

$$G_{n+1}(x) \leq G_n(x) \tag{1. 2. 9}$$

$$G_n(x) = P[E_n(t, x)] \tag{1. 2. 10}$$

and we let

$$A_n(t, x) = \left\{ \begin{array}{l} \text{n occurrences in} \\ \\ \text{the interval } (t, t+x] \end{array} \right\}, \quad x > 0. \tag{1. 2. 11}$$

Observing that, with $h > 0$

$$\{x < L_1(t) \leq x + h\} = \{A_0(t, x) \cap E_1(t + x, h)\} \tag{1. 2. 12}$$

we then have

$$G_1(x + h) - G_1(x) = P[A_0(t, x) \cap E_1(t + x, h)] . \tag{1. 2. 13}$$

In the same way, from

$$\{x < L_n(t) \leq x + h\} = \bigcup_{k=0}^{n-1} \{A_k(t, x) \cap E_{n-k}(t + x, h)\} \tag{1. 2. 14}$$

we obtain, the $A_k$ being disjoint,

$$G_n(x + h) - G_n(x) = \sum_{k=0}^{n-1} P[A_k(t, x) \cap E_{n-k}(t + x, h)] . \tag{1. 2. 15}$$

Furthermore, denoting

$$S_n(x) = \sum_{j=1}^{n} G_j(x) \tag{1. 2. 16}$$

we obtain

$$S_n(x+h) - S_n(x) = \sum_{j=1}^{n}\sum_{k=0}^{j-1} P[A_k(t,x) \cap E_{j-k}(t+x,h)]$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{j} P[A_{j-k}(t,x) \cap E_k(t+x,h)]$$

$$= \sum_{k=1}^{n}\sum_{j=k}^{n} P[A_{j-k}(t,x) \cap E_k(t+x,h)] . \qquad (1.2.17)$$

<u>Properties of $G_1(x)$</u>

For $x_1 \leq x_2$ and any $h > 0$, we have

$$\{A_0(t, x_2) \cap E_1(t + x_2, h\} \subset \{A_0(t + x_2 - x_1, x_1) \cap E_1(t + x_2, h)\} \qquad (1.2.18)$$

hence:

$$G_1(x_2 + h) - G_1(x_2) \leq G_1(x_1 + h) - G_1(x_1) \qquad (1.2.19)$$

$$x_2 \geq x_1 \quad , h > 0$$

In particular, observe that

$$G_1(x+h) - G_1(x) \leq G_1(\mathbf{x}) - G_1(x-h) \leq G_1(x-h) - G_1(x\ 2h) \qquad (1.2.20)$$

$$x \geq 2h , \quad h > 0$$

so that letting $h \to 0$, we obtain

$$G_1(x_+) = G_1(x) = G_1(x_-) \quad , \ x > 0 \qquad (1.2.21)$$

thus showing that $G_1(x)$ is continuous at every $x > 0$.

Moreover, $G_1(x)$ is absolutely continuous in every closed set not containing the origin. Let $\sigma$, v, h, $h_j (j = 1, \ldots n)$ be positive numbers such that

$$v = \sigma - h > 0 \qquad (1.2.22)$$

$$\sum_{j=1}^{n} h_j = h \qquad (1.2.23)$$

and assume that

$$\sigma \leq x_1 < x_2 < x_k < < x_n .$$

Using (1. 2. 19), we can write

$$G_1(x_1 + h_1) - G_1(x_1) \leq G_1(v + h_1) - G_1(v) \tag{1. 2. 24}$$

$$G_1(x_k + h_k) - G_1(x_k) \leq G_1(v + \sum_{j=0}^{k} h_j) - G_1(v + \sum_{j=0}^{k-1} h_j) \tag{1. 2. 25}$$

where we take $h_0 = 0$.

Observing that

$$\sum_{k=1}^{n} [G_1(v + \sum_{j=0}^{k} h_j) - G_1(v + \sum_{j=0}^{k-1} h_j)]$$

$$= G_1(v + \sum_{j=0}^{n} h_j) - G_1(v)$$
$$= G_1(\sigma) - G_1(\sigma - h) \tag{1. 2. 26}$$

we can write

$$\sum_{k=1}^{n}[G_1(x_k + h_k) - G_1(x_k)] \leq G_1(\sigma) - G_1(\sigma - h) \tag{1. 2. 27}$$

By letting $h \to 0$, the right-hand side of this equation tends to zero ($\sigma > 0$ is a point of continuity for $G_1(x)$ ), thereby proving the assertion.

It follows that for $x \geq \sigma > 0$

$$G_1(x) - G_1(\sigma) = \int_{\sigma}^{x} g_1(\zeta)d\zeta \tag{1. 2. 28}$$

where

$$g_1(\zeta) = G_1'(\zeta) \quad \text{a. e.} \tag{1. 2. 29}$$

Letting $\sigma \to 0$, and $g_1(\zeta)$ being an integrable function, we obtain

$$G_1(x) = G_1(0_+) + \int_{0}^{x} g_1(\zeta)d\zeta \tag{1. 2. 30}$$

In addition, in view of (1. 2. 19) we observe that

$$g_1(x_2) \le g_1(x_1) \quad \text{a. e.} \tag{1. 2. 31}$$

$$\text{for } x_2 \ge x_1 \quad .$$

In other words, $g_1(\zeta)$ is a monotone non-increasing function. As a consequence (ref Z.1, Vol. I or H. 1), $G_1(x)$ is concave (or convex upwards) and it satisfies the simplified Jensen inequality,

$$G_1[\alpha x + (1-\alpha)y] \ge \alpha G_1(x) + (1-\alpha)G_1(y) \tag{1. 2. 32}$$

$$0 \le \alpha \le 1 , \quad x > 0 \quad , \quad y > 0 .$$

In particular, letting $y \to 0$

$$G_1(\alpha x) \ge \alpha G_1(x) + (1-\alpha)G_1(0_+) \tag{1. 2. 32'}$$

$$0 < \alpha \le 1 , \quad x > 0$$

and a fortiori

$$G_1(\alpha x) \ge \alpha G_1(x) \tag{1. 2. 33}$$

$$0 < \alpha \le 1 , \quad x > 0 .$$

Properties of $S_n(x)$ and $G_n(x)$

For $x_1 \le x_2$ and any $h > 0$, we observe that

$$\bigcup_{j=k}^{n} \{A_{j-k}(t, x_2) \cap E_k(t + x_2, h)\}$$

$$\subset \bigcup_{j=k}^{n} \{A_{j-k}(t + x_2 - x_1, x_1) \cap E_k(t + x_2, h)\} \tag{1. 2. 34}$$

so that using (1. 2. 17), we obtain

$$S_n(x_2 + h) - S_n(x_2) \le S_n(x_1 + h) - S_n(x_1) \tag{1. 2. 35}$$

$$x_2 \ge x_1 \quad , \quad h > 0 .$$

The situation is the same as in equation (1.2.19), hence similar results follow: $S_n(x)$ is concave and

$$S_n(x) = S_n(0_+) + \int_0^x s_n(\zeta)\,d\zeta \tag{1.2.36}$$

where

$$s_n(\zeta) = S_n'(\zeta) \quad \text{a.e.} \quad . \tag{1.2.37}$$

From

$$G_n(x) = S_n(x) - S_{n-1}(x) \tag{1.2.38}$$

we conclude that each $G_n(x)$ is absolutely continuous for $x > 0$ and that

$$G_n(x) = G_n(0_+) + \int_0^x g_n(\zeta)\,d\zeta \tag{1.2.39}$$

where

$$g_n(\zeta) = G_n'(\zeta) \quad \text{a.e.} \quad . \tag{1.2.40}$$

Notice

$$g_n(\zeta) = s_n(\zeta) - s_{n-1}(\zeta) \tag{1.2.41}$$

so that $g_n(\zeta)$ is a function of bounded variation in every closed set not containing the origin (difference of two bounded non-increasing functions).

1.2.4.2 Higher order properties

As an example, consider the joint distribution function

$$G_{1,2}(x, y) = P[L_1(t) \leq x \cap L_2(t) \leq y]. \tag{1.2.4.2}$$

Observe that

$$G_{1,2}(x+h_1, y+h_2) - G_{1,2}(x, y) \leq \{G_1(x+h_1) - G_1(x)\} + \{G_2(y+h_2) - G_2(y)\}, \tag{1.2.43}$$

$$h_1,\ h_2 > 0,$$

and consequently that $G_{1,2}(x, y)$ is continuous for $x, y > 0$. But $G_{1,2}(x, y)$ may not be absolutely continuous; an example is provided by the stationary periodic point process (defined in section 1.4.1).

Next, interpreting equation (1.2.34) and choosing $k = 1$, $n = 2$, we obtain

$$\{G_1(x_2 + h) - G_1(x_2)\} + \{G_{1,2}(x_2, x_2 + h) - G_{1,2}(x_2, x_2)\}$$

$$\leq \{G_1(x_1 + h) - G_1(x_1)\} + \{G_{1,2}(x_1, x_1 + h) - G_{1,2}(x_1, x_1)\} \qquad (1.2.44)$$

$$x_2 \geq x_1 \ , \ h > 0$$

which is somewhat similar to (1.2.35).

Following the same line of approach, one obtains similar properties for higher-order distributions functions.

## 1.3 The number of occurrences generated by a stationary point process in a given time interval

Because of stationarity, equations (1.1.15) and (1.1.16) do not depend on t. Therefore

$$p(0, \tau) = 1 - G_1(\tau) \qquad (1.3.1)$$

$$p(n, \tau) = G_n(\tau) - G_{n+1}(\tau) \qquad (1.3.2)$$

where

$$p(n, \tau) = P[N(t, \tau) = n] \ . \qquad (1.3.3)$$

Whenever there is no ambiguity, the notation $N(\tau)$ will be used instead of $N(t, \tau)$.

The following lemmas will be used later on.

Lemma 1: $G_{2n}(2\tau) \leq 2G_n(\tau)$ . (1.3.4)

Using (1.2.1), (1.2.10) and observing that $E_{2n}(t, 2\tau) \subset E_n(t, \tau) \cup E_n(t+\tau, \tau)$ we arrive at (1.3.4).

Lemma 2: If $\lim_{n\to\infty} n^{\lambda} G_{2n}(\tau) = 0$ , $\lambda \geq 0$ then $\lim_{n\to\infty} n^{\lambda} G_n(\tau) = 0$. (1.3.5)

It is sufficient to show that (1.3.5) holds when n is odd; in view of (1.2.9), we observe that

$$(2k + 1)^{\lambda} G_{2k+1}(\tau) \leq 3^{\lambda} k^{\lambda} G_{2k}(\tau)$$

from which the assertion follows.

Lemma 3: If, for $\tau > 0$, $h > 0$, we let

$$V_n(\tau, h) = \frac{S_n(\tau + h) - S_n(\tau)}{h} \tag{1.3.6}$$

where $S_n(\tau)$ is defined by (1.2.16), (1.2.36), then $V_n(\tau, h)$ is monotone non-decreasing in the three variables as $n$ increases, $\tau$ decreases and h decreases.

Let $\tau_1, \tau_2, h_1, h_2$ be positive numbers such that $\tau_1 \leq \tau_2$, $h_1 \leq h_2$. Writing

$$S_n(\tau_2 + h_1) = S_n\left[\frac{h_2 - h_1}{h_2}\tau_2 + \frac{h_1}{h_2}(\tau_2 + h_2)\right] ,$$

the Jensen inequality (1.2.32) gives

$$S_n(\tau_2 + h_1) \geq \frac{h_2 - h_1}{h_2} S_n(\tau_2) + \frac{h_1}{h_2} S_n(\tau_2 + h_2)$$

or

$$\frac{S_n(\tau_2 + h_2) - S_n(\tau_2)}{h_2} \leq \frac{S_n(\tau_2 + h_1) - S_n(\tau_2)}{h_1} .$$

Because of (1.2.35)

$$\frac{S_n(\tau_2 + h_2) - S_n(\tau_2)}{h_2} \leq \frac{S_n(\tau_1 + h_1) - S_n(\tau_1)}{h_1}$$

and *a fortiori*

$$\frac{S_n(\tau_2 + h_2) - S_n(\tau_2)}{h_2} \leq \frac{S_{n+1}(\tau_1 + h_1) - S_{n+1}(\tau_1)}{h_1}$$

in other words

$$V_n(\tau_2, h_2) \leq V_{n+1}(\tau_1, h_1) \tag{1.3.7}$$

$$\tau_1 \leq \tau_2 , \quad h_1 \leq h_2 .$$

In what follows, we consider the behavior of the random variable $N(\tau)$.

<u>Theorem 1:</u>

i) $N(\tau)$ is finite valued if, and only if

$$\lim_{n \to \infty} G_n(\tau) = 0. \tag{1.3.8}$$

ii) If for some value $\tau_0 > 0$

$$\lim_{n \to \infty} G_n(\tau_0) = 0 \tag{1.3.9}$$

then (1.3.8) holds for all $\tau < \infty$.

Proof.

i) From (1.3.1), (1.3.2), we obtain

$$\sum_{k=0}^{n} p(k, \tau) = 1 - G_{n+1}(\tau)$$

and so

$$\sum_{k=0}^{\infty} p(k, \tau) = 1$$

if, and only if

$$\lim_{n \to \infty} G_n(\tau) = 0.$$

ii) In view of

$$G_n(\tau_1) \leq G_n(\tau_2) \quad , \quad \tau_1 \leq \tau_2$$

it will be enough to show that (1.3.9) implies $\lim\limits_{n \to \infty} G_n(2\tau_0) = 0$ (since from there, one shows that for every $K = 1, 2, \ldots$

$$\lim_{n \to \infty} G_n(2^K \tau_0) = 0 \,).$$

Because of lemma 1

$$\lim_{n \to \infty} G_{2n}(2\tau_0) = 0$$

and lemma 2 (with $\lambda = 0$) completes the proof.

Theorem 2:

$$E[N(\tau)^M] < \infty \ , \quad M \in (1, 2, \ldots) \tag{1.3.10}$$

if, and only if

$$\sum_{n=1}^{\infty} [n^M - (n-1)^M] G_n(\tau) < \infty. \tag{1.3.11}$$

Moreover

$$E[N(\tau)^M] = \sum_{n=1}^{\infty} [n^M - (n-1)^M] G_n(\tau) \ . \tag{1.3.12}$$

Proof.

Consider the partial sums

$$U_n = \sum_{k=1}^{n} [k^M - (k-1)^M] \, G_k \tag{1.3.13}$$

$$W_n = \sum_{k=1}^{n} k^M p(k, \tau)$$

$$= \sum_{k=1}^{n} k^M (G_k - G_{k+1}) \tag{1.3.14}$$

where for simplicity we abbreviate $G_k(\tau)$ by $G_k$. Comparing (1.3.13) and (1.3.14) leads to

$$W_n + n^M G_{n+1} = U_n \tag{1.3.15}$$

and a fortiori

$$W_n \le U_n \ . \tag{1.3.16}$$

s) (1.3.11) implies that

$$\lim_{n \to \infty} G_n(\tau) = 0$$

so that $N(\tau)$ is finite valued; moreover, because of (1.3.16)

$$\lim_{n \to \infty} W_n < \infty$$

or

$$E[N(\tau)^M] < \infty.$$

n) Conversely, (1.3.10) implies

$$\lim_{n\to\infty} G_n(\tau) = 0 \qquad \text{(Theorem 1)} \tag{1.3.17}$$

and

$$\sum_{k=1}^{\infty} k^M (G_k - G_{k+1}) < \infty . \tag{1.3.18}$$

Equation (1.3.18) implies that for every arbitrary given $\epsilon > 0$ and all p, we may write

$$0 \le \sum_{k=n}^{n+p} k^M [G_k - G_{k+1}] < \epsilon$$

$$\text{for } n > N(\epsilon)$$

or rearranging the terms

$$0 \le \{n^M[G_n - G_{n+p+1}]\} + \{\sum_{k=n+1}^{n+p} [k^M - (k-1)^M] G_k - [(n+p)^M - n^M] G_{n+p+1}\} < \epsilon$$

$$\text{for } n > N(\epsilon) \tag{1.3.19}$$

In view of (1.2.9), the terms $\{\ \}$ are positive, so that (1.3.19) leads to

$$0 \le n^M [G_n - G_{n+p+1}] < \epsilon \tag{1.3.20}$$

$$\text{for } n > N(\epsilon)$$

Letting $p \to \infty$, and using (1.3.17) we obtain

$$n^M G_n < \epsilon , \quad n > N(\epsilon)$$

or that

$$\lim_{n\to\infty} n^M G_n(\tau) = 0 \tag{1.3.21}$$

As a consequence, from (1.3.15)

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} W_n < \infty$$

or that

$$\sum_{n=1}^{\infty}[n^M - (n-1)^M]G_n(\tau) = E[N(\tau)^M], \qquad (1.3.22)$$

thus completing the proof.

The cases M = 1, M = 2 are of particular interest and deserve special attention.

Theorem 3: If for some $\tau = \tau_0 > 0$

i) $$E[N(\tau_0)] < \infty \qquad (1.3.23)$$

or

ii) $$E[N(\tau_0)^2] < \infty \qquad (1.3.24)$$

then the same properties hold for all $\tau < \infty$.

Proof.

For similar reasons as in part ii) of theorem 1, it will be sufficient to show that these properties hold for $\tau = 2\tau_0$. Two different proofs will be given.

a) From

$$N(t, 2\tau_0) = N(t, \tau_0) + N(t + \tau_0, \tau_0) \qquad (1.3.25)$$

it follows

i) $$E[N(2\tau_0)] = 2E[N(\tau_0)] < \infty \qquad (1.3.26)$$

ii) $$E[N(2\tau_0)^2] = 2E[N(\tau_0)^2] + E[N(t, \tau_0)N(t+\tau_0, \tau_0)]$$

and because of the Schwarz inequality

$$E[N(2\tau_0)^2 \leq 4E[N(\tau_0)^2] < \infty \qquad (1.3.27)$$

thus completing the proof.

b) Using (1. 3. 22) with M = 1, M = 2, we have

$$E[N(\tau_0)] = \sum_{n=1}^{\infty} G_n(\tau_0) \tag{1.3.28}$$

$$E[N(\tau_0)^2] = \sum_{n=1}^{\infty} (2n - 1)\, G_n(\tau_0) \tag{1.3.29}$$

i) Consider the partial sum

$$S_n(\tau) = \sum_{k=1}^{n} G_k(\tau) \;\; .$$

$S_n(\tau)$ is concave and satisfies an inequality similar to (1. 2. 33) so that

$$S_n(2\tau_0) \le 2S_n(\tau_0)$$

and consequently

$$E[N(2\tau_0)] = \lim_{n\to\infty} \;\; S_n(2\tau_0) \; < \infty \;\; .$$

ii) Let

$$V_{2n+1}(2\tau_0) = \sum_{k=1}^{2n+1} (2k - 1) G_k(2\tau_0) \; . \tag{1.3.30}$$

In view of (1. 2. 9), we obtain

$$V_{2n+1}(2\tau_0) \le G_1(2\tau_0) + 8\sum_{k=1}^{n} k\, G_{2k}(2\tau_0)$$

and, a fortiori, from lemma 1

$$V_{2n+1}(2\tau_0) \le G_1(2\tau_0) + 16\sum_{k=1}^{n} k\, G_k(\tau_0).$$

Because of (1. 3. 24), (1. 3. 29) the right-hand side converges as $n \to \infty$; hence

$$\lim_{n\to\infty} \;\; V_{2n+1}(2\tau_0) \; < \infty$$

and as a result

$$E[N(2\tau_0)^2] < \infty \; .$$

Theorem 4:

$$E[N(\tau)] < \infty \tag{1. 3. 31}$$

if, and only if

$$\beta = \sum_{n=1}^{\infty} \lim_{h \to 0_+} \frac{G_n(h)}{h} < \infty . \tag{1. 3. 32}$$

Moreover

$$E[N(\tau)] = \beta \tau \tag{1. 3. 33}$$

$$\sum_{n=1}^{\infty} g_n(x) = \beta \qquad \text{a. e.} \tag{1. 3. 34}$$

where the $g_n(x)$ are density functions defined as

$$g_n(x) = G_n'(x) \qquad \text{a. e.} \quad . \tag{1. 3. 35}$$

Finally, we can write

$$g_1(x) = \beta[1 - F_1(x)] \tag{1. 3. 36}$$

$$g_n(x) = \beta[F_{n-1}(x) - F_n(x)] \tag{1. 3. 37}$$

where the $F_n(x)$ are distribution functions having the following properties:

$$\int_0^{\infty} x \, dF_n(x) = \frac{n}{\beta} \tag{1. 3. 38}$$

$$\lim_{n \to \infty} F_n(x) = 0 \tag{1. 3. 39}$$

$$\lim_{x \to 0_+} F_n(x) = 1 - \frac{1}{\beta} \sum_{k=1}^{n} \lim_{h \to 0_+} \frac{G_k(h)}{h} . \tag{1. 3. 40}$$

Proof.

n) From (1. 3. 28), we have

$$E[N(\tau)] = \sum_{n=1}^{\infty} G_n(\tau) < \infty . \tag{1. 3. 41}$$

This convergence is uniform for $\tau \leq \tau_0$ for arbitrary $\tau_0 < \infty$. In fact

$$G_n(\tau) \leq G_n(\tau_0) \quad , \quad \tau \leq \tau_0 \, , \tag{1. 3. 42}$$

$$\sum_{n=1}^{\infty} G_n(\tau_0) = E[N(\tau_0)] < \infty \tag{1. 3. 43}$$

from which (Weierstrass' M-test) (1. 3. 41) converges uniformly for $\tau \leq \tau_0$. On the other hand, we have seen that the functions $G_n(\tau)$ are continuous for $\tau > 0$. It follows (ref K. 2, page 339) that $E[N(\tau)]$ is continuous for $\tau > 0$. Moreover

$$N(t, \tau_1 + \tau_2) = N(t, \tau_1) + N(t + \tau_1, \tau_2), \quad \tau_1 > 0, \ \tau_2 > 0$$

leads to the functional equation

$$E[N(\tau_1 + \tau_2)] = E[N(\tau_1)] + E[N\tau_2)]$$

which admits only one continuous solution (it is interesting to note that there are infinitely many discontinuous solutions; see Hamel functions in ref W. 4), namely

$$E[N(\tau)] = \sum_{n=1}^{\infty} G_n(\tau) = \beta\tau \tag{1. 3. 44}$$

where $\beta$ is some positive constant. The convergence being uniform, it follows

$$\sum_{n=1}^{\infty} \lim_{\tau \to 0} G_n(\tau) = \lim_{\tau \to 0} \sum_{n=1}^{\infty} G_n(\tau). \tag{1. 3. 45}$$

Thus

$$\sum_{n=1}^{\infty} G_n(0_+) = 0 \tag{1. 3. 46}$$

or what is equivalent, in view of (1. 2. 9),

$$G_1(0_+) = 0 \, . \tag{1. 3. 47}$$

Next, as a result of lemma 3, we can apply convergence theorems to monotone functions or series (ref H. 3, pages 414-5 vol I, page 47 vol II) so that

$$\lim_{n\to\infty} \lim_{h\to\infty} \lim_{\tau\to\infty} V_n(\tau, h) = \lim_{h\to 0} \lim_{n\to\infty} \lim_{\tau\to 0} V_n(\tau, h) \tag{1. 3. 48}$$

or

$$\lim_{n\to\infty}\lim_{h\to 0}\frac{S_n(h)}{h} = \lim_{h\to 0}\lim_{n\to\infty}\frac{S_n(h)}{h}$$

and hence

$$\sum_{n=1}^{\infty}\lim_{h\to 0}\frac{G_n(h)}{h} = \lim_{h\to 0}\sum_{n=1}^{\infty}\frac{G_n(h)}{h} = \beta. \tag{1.3.49}$$

This result may also be obtained by using uniform convergence properties as in (1.3.45).

s) We know that $S_n(\tau)$ is concave and satisfies an inequality similar to (1.2.33), consequently

$$\frac{S_n(\tau)}{\tau} \le \frac{S_n(h)}{h}\ , \qquad h \le \tau\ . \tag{1.3.50}$$

In particular

$$\frac{S_n(\tau)}{\tau} \le \lim_{h\to 0}\frac{S_n(h)}{h}\ ,$$

or

$$\frac{1}{\tau}\sum_{k=1}^{n} G_k(\tau) \le \sum_{k=1}^{n}\lim_{h\to 0}\frac{G_n(h)}{h} \tag{1.3.51}$$

Letting $n \to \infty$, and because of (1.3.32) we conclude that

$$E[N(\tau)] = \sum_{k=1}^{\infty} G_k(\tau) < \infty$$

thus proving (1.3.31).

Next, we consider

$$\sum_{n=1}^{\infty} G_n(x) = \beta x\ . \tag{1.3.52}$$

Since the $G_n(x)$ are monotone non-decreasing, Fubini's derivation theorem applies (ref R. 2, page 11), that is derivation term by term is permissible a. e., and hence

$$\sum_{n=1}^{\infty} g_n(x) = \beta \qquad \text{a. e.}, \tag{1.3.53}$$

where

$$g_n(x) = G_n'(x) \quad \text{a. e.}$$

Because of (1. 2. 30), (1. 2. 39), (1. 3. 47) we obtain

$$G_1(\tau) = \int_0^\tau g_1(x)\,dx \tag{1. 3. 54}$$

$$G_n(\tau) = \int_0^\tau g_n(x)\,dx \tag{1. 3. 55}$$

thus showing that the $g_n(x)$ are density functions.

Let us first consider $g_1(x)$. Because of (1. 2. 31) and (1. 3. 53), we may let

$$g_1(x) = \beta[1 - F_1(x)], \quad x > 0 \tag{1. 3. 56}$$

and observing that the function $F_1(x)$ is non-decreasing, positive and bounded by one. More precisely, we want to show that $F_1(x)$ may be considered as a distribution function having a positive finite mean: the fact that

$$\int_0^\infty g_1(x)\,dx = 1 \tag{1. 3. 57}$$

implies that for arbitrary $\epsilon > 0$, we may write

$$0 \le \int_{x_1}^{x_2} g_1(x)\,dx < \epsilon \qquad \text{for } x_2 \ge x_1 > x(\epsilon) \tag{1. 3. 58}$$

and a fortiori

$$0 \le (x_2 - x_1)\,g_1(x_2) < \epsilon \qquad \text{for } x_1 \ge x_1 > X(\epsilon) \tag{1. 3. 59}$$

since $g_1(x)$ is non-increasing.

Letting $x_2 - x_1 = 1$, we see that

$$0 \leq g_1(x_2) < \epsilon$$

$$\text{for } x_2 > X(\epsilon) + 1$$

so that

$$\lim_{x \to \infty} g_1(x) = 0$$

and, from (1. 3. 56),

$$\lim_{x \to \infty} F_1(x) = 1. \tag{1. 3. 60}$$

Next, taking $x_2 = 2x_1$, we see that

$$0 \leq x_2 g_1(x_2) < 2\epsilon$$

$$\text{for } x_2 > 2X(\epsilon)$$

so that

$$\lim_{x \to \infty} x\, g_1(x) = 0$$

and an integration by parts of (1. 3. 57) gives

$$\int_0^\infty x\, dF_1(x) = \frac{1}{\beta} . \tag{1. 3. 61}$$

As a result of lemma 3, we can write

$$\lim_{h \to 0} \lim_{\tau \to 0} V_1(\tau, h) = \lim_{\tau \to 0} \lim_{h \to 0} V_1(\tau, h)$$

or

$$\lim_{h \to 0} \frac{G_1(h)}{h} = g_1(0_+) \quad ,$$

that is

$$F_1(0_+) = 1 - \frac{1}{\beta} \lim_{h \to 0} \frac{G_1(h)}{h} \quad . \tag{1. 3. 62}$$

We next consider the more general case. Using (1. 2. 37), we let

$$s_n(x) = \beta[1 - F_n(x)] \tag{1. 3. 63}$$

and since $s_n(x)$ is non-increasing we have the same situation as with $g_1(x)$.

We are thus led to similar results, namely

$$\lim_{x\to\infty} F_n(x) = 1 \tag{1.3.64}$$

and

$$\int_0^\infty x\, dF_n(x) = \frac{n}{\beta} \;. \tag{1.3.65}$$

Moreover

$$F_n(0_+) = 1 - \frac{1}{\beta}\sum_{k=1}^{n} \lim_{h\to 0} \frac{G_k(h)}{h} \;. \tag{1.3.66}$$

From

$$s_n(x) = \sum_{k=1}^{n} g_k(x) \tag{1.3.67}$$

we obtain

$$g_n(x) = \beta[F_{n-1}(x) - F_n(x)] \tag{1.3.68}$$

and observe that

$$F_{n+1}(x) \leq F_n(x) \;. \tag{1.3.69}$$

Finally, because of (1.3.53), we obtain

$$\lim_{n\to\infty} F_n(x) = 0 \quad , \tag{1.3.70}$$

thus completing the proof.

An intuitive interpretation of the $F_n(x)$ will be given in the appendix B.

Theorem 5: If

$$E[N(\tau)^2] < \infty \tag{1.3.71}$$

then

$$H(x) = \sum_{n=1}^{\infty} F_n(x) < \infty \tag{1.3.72}$$

where the $F_n(x)$ are the distribution functions defined in theorem 4.

Moreover

$$E[N(\tau)^2] = \beta\tau + 2\beta\int_0^\tau H(x)\, dx \tag{1.3.73}$$

and

$$\sum_{n=1}^{\infty} f_n(x) < \infty \qquad \text{a. e.} \tag{1.3.74}$$

where

$$f_n(x) = F_n'(x) \qquad \text{a. e.} \tag{1.3.75}$$

Proof.

From (1. 3. 29), we have

$$\sum_{n=1}^{\infty} (2n-1) G_n(\tau) = E[N(\tau)^2] < \infty \tag{1.3.76}$$

or

$$\sum_{n=1}^{\infty} \int_0^{\tau} (2n-1) g_n(x)\,dx = E[N(\tau)^2] < \infty .$$

Combining Beppo-Levi's and Fubini's derivation theorems (ref R. 2 pages 11, 35), we obtain

$$\sum_{n=1}^{\infty} (2n-1) g_n(x) = e(x) < \infty \qquad \text{a. e.}, \tag{1.3.77}$$

where

$$e(x) = \frac{d}{dx}\left\{E[N(x)^2]\right\} \qquad \text{a. e.},$$

and in addition that

$$E[N(\tau)^2] = \int_0^{\tau} e(x)\,dx \tag{1.3.78}$$

thus showing that the function $E[N(\tau)^2]$ is absolutely continuous for $\tau < \infty$.

On the other hand, using (1. 3. 36), (1. 3. 37), we can write

$$\sum_{n=1}^{\infty} (2n-1) g_n(x) = \beta[1 - F_1(x) + \sum_{n=2}^{\infty} (2n-1)\{F_{n-1}(x) - F_n(x)\}] \tag{1.3.79}$$

Using properties (1. 3. 69) and (1. 3. 70) and the hypothesis (1. 3. 71) it may be shown (as in theorem 2), that

$$\lim_{n\to\infty} n F_n(x) = 0 \tag{1.3.80}$$

and

$$\sum_{n=1}^{\infty} (2n - 1)\, g_n(x) = \beta[1 + 2 \sum_{n=1}^{\infty} F_n(x)] \tag{1. 3. 81}$$

so that

$$H(x) = \sum_{n=1}^{\infty} F_n(x) < \infty. \tag{1. 3. 82}$$

This function H(x) appears similar to the so-called "renewal function" defined in ref M. 1, pages 364-5.

Using (1. 3. 78) and (1. 3. 81), we arrive at

$$E[N(\tau)^2] = \beta\tau + 2\beta \int_0^{\tau} H(x)\, dx \tag{1. 3. 83}$$

which is analogous to equation (3. 4), page 370, ref M. 1.

Applying once more Fubini's derivation theorem to (1. 3. 82), we obtain

$$\sum_{n=1}^{\infty} f_n(x) = h(x) < \infty \qquad \text{a. e.} \tag{1. 3. 84}$$

where

$$f_n(x) = F'_n(x)\,, \quad h(x) = H'(x) \qquad \text{a. e.}$$

Theorem 6:

i)

$$E[N(\tau)^M] < \infty \tag{1. 3. 85}$$

for all $M = 1, 2, \ldots$

if, and only if

$$\lim_{n\to\infty} n^K G_n(\tau) = 0 \tag{1. 3. 86}$$

for all $K = 1, 2, \ldots$ .

ii) If for some value $\tau_0 > 0$

$$\lim_{n\to\infty} n^K G_n(\tau_0) = 0, \quad K = 1, 2, \ldots \tag{1. 3. 87}$$

then (1. 3. 86) holds for all $\tau < \infty$.

Proof.

i) The necessary part was already seen in (1. 3. 21). For the converse we have to show that

$$E[N(\tau)^M] = \sum_{n=1}^{\infty} n^M p(n, \tau) < \infty$$

Choosing K = M + 2, (1. 3. 86) implies that for arbitrary $\epsilon > 0$

$$n^K G_n(\tau) < \epsilon, \quad n \geq N(\epsilon, K) \tag{1. 3. 88}$$

and consequently that

$$\sum_{n=N(\epsilon, K)}^{\infty} n^M p(n, \tau) < \sum_{n=N(\epsilon, K)}^{\infty} n^M G_n(\tau)$$

$$< \sum_{n=N(\epsilon, k)}^{\infty} \frac{\epsilon}{n^2}$$

$$< \epsilon \sum_{n=1}^{\infty} \frac{1}{n^2} = \epsilon \frac{\pi^2}{6} .$$

ii) It is sufficient to show that (1. 3. 87) implies

$$\lim_{n\to\infty} n^K G_n(2\tau_0) = 0, \tag{1. 3. 89}$$

which is obtained by applying successively lemma 1 and lemma 2 to $n^K G_n(\tau_0)$.

## Conclusions

For stationary point processes we have observed the following points of interest:

i) The distribution functions $G_n(\tau)$ are absolutely continuous, except perhaps at the origin. In addition, for every n, the partial sum $S_n(\tau) = \sum_{k=1}^{n} G_k(\tau)$ is a concave function.

ii) If for some $\tau = \tau_0 > 0$

$$E[N(\tau_0)] < \infty \tag{1. 3. 90}$$

then the same holds for all $\tau < \infty$, which further implies that the $G_n(\tau)$ are absolutely continuous everywhere. Then

$$\sum_{n=1}^{\infty} G_n(\tau) = \beta\tau ,$$

where $\beta$ is a positive constant (Notice that the case $\beta = 0$ corresponds to an empty point process). Moreover, the density functions $g_n(x) = G_n'(x)$ a. e. have a noteworthy structure in terms of distribution functions $\mathbf{F}_n(x)$ having certain analogies with the $G_n(x)$ [compare (1. 3. 69), (1. 3. 70) with (1. 2. 9), (1. 3. 8)]. It was also shown that (1. 3. 90) is equivalent to

$$\sum_{n=1}^{\infty} \lim_{h\to 0} \frac{G_n(h)}{h} < \infty. \qquad (1.3.91)$$

iii) If for some $\tau = \tau_0 > 0$

$$E[N(\tau_0)^2] < \infty \qquad (1.3.92)$$

then the same property holds for all $\tau < \infty$ (Theorem 3). In addition, the function $E[N(\tau)^2]$ is absolutely continuous for $\tau < \infty$ and is entirely expressed in terms of the $\mathbf{F}_n(x)$, as was seen in theorem 5 (similarly, one can express higher order moments $E[N(\tau)^M]$, $M \geq 3$ in terms of the $\mathbf{F}_n(x)$ ).

iv) The property

$$\lim_{n\to\infty} n^K G_n(\tau_0) = 0, \quad \tau_0 > 0, \; K = 1, 2, \ldots$$

is equivalent to the existence of all moments.

Since we are dealing with non-negative random variables, it will be useful to rewrite (1. 3. 1), (1. 3. 2), (1. 3. 36), (1. 3. 37) in terms of Laplace-Stieltjes transforms (ref W. 1). Letting

$$g_n^*(s) = \int_{0-}^{\infty} e^{-sx}\, dG_n(x) \qquad (1.3.93)$$

$$p^*(n, s) = \int_{0}^{\infty} e^{-s\tau} p(n, \tau)\, d\tau \qquad (1.3.94)$$

$$f_n^*(s) = \int_{0-}^{\infty} e^{-sx} dF_n(x) \qquad (1.3.95)$$

we obtain

$$p^*(0, s) = \frac{1 - g_1^*(s)}{s} \tag{1.3.96}$$

$$p^*(n, s) = \frac{g_n^*(s) - g_{n+1}^*(s)}{s} \tag{1.3.97}$$

and

$$g_1^*(s) = \beta \frac{1 - f_1^*(s)}{s} \tag{1.3.98}$$

$$g_n^*(s) = \beta \frac{f_{n-1}^*(s) - f_n^*(s)}{s} . \tag{1.3.99}$$

## 1.4 Examples of stationary point processes

### 1.4.1 The periodic point process

This well-known stationary point process is defined as

$$L_n = (n - 1)T + \alpha \quad , \quad n = 1, 2, \ldots \tag{1.4.1}$$

where T is a positive constant called the period and where $\alpha$ is a random variable uniformly distributed in the interval $(0, T]$. Observe that

$$G_n(\tau) = \begin{cases} 0 & , \quad \tau \le (n-1)T \\ \frac{1}{T}[\tau - (n-1)T] & , \quad (n-1)T < \tau \le nT \\ 1 & , \quad \tau > nT \end{cases} \tag{1.4.2}$$

or

$$g_n^*(s) = \frac{1}{T} \frac{1 - e^{-sT}}{s} e^{-s(n-1)T} \tag{1.4.3}$$

It follows from theorem 6, that $N(\tau)$ has finite moments of all orders as expected.

### 1.4.2 The Poisson point process

This symmetric stationary point process is also well-known (ref F. 1, P. 2, T. 1), and is defined as

$$L_n = L_1 + x_1 + \cdots + x_{n-1}$$

where $L_1, x_1, \ldots x_{n-1}$ are independent random variables, identically distributed with density function

$$g_1(x) = f_1(x) = \beta e^{-\beta x} \quad , \; \beta > 0 \tag{1. 4. 5}$$

so that

$$g_n^*(s) = E[e^{-sL_n}]$$

$$= \left\{ \frac{\beta}{\beta + s} \right\}^n \quad , \tag{1. 4. 6}$$

or that

$$g_n(\tau) = \beta \frac{1}{(n-1)!} (\beta\tau)^{n-1} e^{-\beta\tau} \; . \tag{1. 4. 7}$$

Using theorem 6, one shows that $N(\tau)$, $\tau < \infty$ has finite moments of all orders. In addition, we observe that

$$\sum_{n=1}^{\infty} g_n(\tau) = \beta$$

$$E[N(\tau)] = \beta\tau$$

and (1. 3. 76) leads to

$$E[N(\tau)^2] = \beta\tau + \beta^2\tau^2 \; . \tag{1. 4. 8}$$

The Poisson point process is often called a "purely" random process or a process with "no memory" since it can be shown that the numbers of occurrences in non-overlapping intervals are independent. The instants of emission of the electrons in a vacuum tube constitute a Poisson point process.

1. 4. 3 The zero-crossings of the Ornstein-Uhlenbeck process

In this example the stationary point process $\{t_n\}$ is generated by the zero-crossings of a symmetric stationary Gaussian Markov process x(t), i. e.:

$$t_n \in \{x(t) = 0\} \; . \tag{1. 4. 9}$$

This point process has been studied by various authors (see the list of references in S. 3, M. 1, L. 3) and it has been shown that

$$G_1(\tau) = 1 - \frac{2}{\pi} \text{ Arc sin } e^{-\tau} \; . \tag{1. 4. 10}$$

It follows that

$$\lim_{h \to 0+} \frac{G_1(h)}{h} = \infty \tag{1.4.11}$$

and as a result of theorem 4, we obtain

$$E[N(\tau)] = \infty . \tag{1.4.12}$$

It is of interest to observe that this result did not require the knowledge of all the $G_n(\tau)$, $n \geq 2$. In fact, (1.4.12) is well-known and has been usually derived by different methods.

1.4.4 A stationary point process with periodic limit points

On the interval $[k, k+1]$, $k = 0, \pm 1, \pm 2$, we define the set of points

$$T_k = \{t_m^k\} \tag{1.4.13}$$

where

$$t_m^k = k - \frac{1}{m}, \quad m = 2, 3, \ldots \quad . \tag{1.4.14}$$

Denoting by $\alpha$ a random variable uniformly distributed between zero and one, the point process generated from the set

$$T = \bigcup_{k=-\infty}^{\infty} T_k + \alpha \tag{1.4.15}$$

is stationary. Then, we obtain

$$G_n(1 + \epsilon) = 1, \ \epsilon > 0, \quad n = 1, 2, \ldots \tag{1.4.16}$$

and theorem 1 leads to

$$N(1 + \epsilon) = \infty , \tag{1.4.17}$$

the result naturally expected. We should note here that the behavior of $G_1(\tau)$ for $\tau \leq \frac{1}{2}$ requires a more involved calculation. It has been found that

$$\lim_{h \to 0} \frac{G_1(h)}{h} = \infty ; \tag{1.4.18}$$

and, in view of theorem 4, we conclude

$$E[N(\tau)] < \infty ,$$

as it should be.

1.4.5 Stationary point processes with skips

Let us consider a stationary point process $\{t_n\}$ where the $t_n$ denote the intended instants for occurrence of events. Suppose that at each scheduled instant $t_n$, the probability of skipping the event is $q < 1$, and assume that all the skips occur independently. The remaining points, $\{t'_n\}$, constitute a new stationary point process which is called a stationary point process with skips. We shall now relate the two point processes.

Using the usual notations, we can write

$$P[L'_1 \le \tau] = \sum_{n=1}^{\infty} P[L_n \le \tau]\, P[t'_1 = t_n] \tag{1.4.19}$$

and since

$$P[t'_1 = t_n] = q^{n-1}(1 - q) \tag{1.4.20}$$

we obtain

$$G'_1(\tau) = (1 - q) \sum_{n=0} q^n G^*_{n+1}(\tau) \tag{1.4.21}$$

or

$$g'^*_1(s) = (1 - q) \sum_{n=0} q^n\, g^*_{n+1}(s) \tag{1.4.22}$$

thus giving the distribution function for the first passage time in the point process with skips. Observe that

$$\lim_{\tau\to\infty} G'_1(\tau) = (1 - q) \lim_{\tau\to\infty} \lim_{k\to\infty} \sum_{n=0}^{k} q^n G_{n+1}(\tau)$$

$$= (1 - q) \lim_{k\to\infty} \lim_{\tau\to\infty} \sum_{n=0}^{k} q^n G_{n+1}(\tau)$$

$$= (1 - q) \sum_{n=0}^{\infty} q^n = 1 ,$$

as it should.

More generally we have

$$P[L_n' \le \tau] = \sum_{m=0}^{\infty} P[L_{n+m} \le \tau]\, P\left[t_n' = t_{n+m} \bigcap \begin{matrix}\text{m skips in}\\ \text{m+n-1 trials}\end{matrix}\right] \qquad (1.4.23)$$

and since

$$P\left[t_n' = t_{n+m} \bigcap \begin{matrix}\text{m skips in}\\ \text{n+m-1 trials}\end{matrix}\right] = (1-q)\, C_{n+m-1}^{n-1} (1-q)^{n-1} q^m \qquad (1.4.24)$$

where

$$C_n^k = \frac{n!}{k!(n-k)!} \qquad (1.4.25)$$

it follows that

$$G_n'(\tau) = (1-q)^n \sum_{m=0}^{\infty} q^m C_{n+m-1}^{n-1} G_{n+m}(\tau) \qquad (1.4.26)$$

or

$$g_n'^*(s) = (1-q)^n \sum_{m=0}^{\infty} q^m C_{n+m-1}^{n-1} g_{n+m}^*(s) \;. \qquad (1.4.27)$$

Higher-order distribution functions can be calculated in a similar way. Furthermore, if $L_1, x_1, \ldots x_n$ ($x_n = L_{n+1} - L_n$) are independent random variables such that all the $x_n$ are identically distributed according to

$$f_1^*(s) = E[e^{-sx_n}] \,, \qquad (1.4.28)$$

then the $L_1'$, $x_1'$, $\ldots x_n'$ are independent and the $x_n'$ are identically distributed, and we have

$$g_1'^*(s) = (1-q)\frac{g_1^*(s)}{1 - q f_1^*(s)} \qquad (1.4.29)$$

$$f_1'^*(s) = E[e^{-sx_n'}] = (1-q)\frac{f_1^*(s)}{1 - q f_1^*(s)} \qquad (1.4.30)$$

<u>Example 1: The Poisson point process with skips</u>

Applying (1.4.5) to the preceding equations, we obtain

$$g_1'^*(s) = f_1'^*(s) = \frac{(1 - q)\beta}{s + (1 - q)\beta} \tag{1.4.31}$$

showing that the Poisson point proc ess with skips is again a Poisson point process with parameter $(1 - q)\beta$. This result is well-known (ref P. 2).

Example 2: The periodic point process with skips

Using (1.4.1), (1.4.3), equation (1.4.29), (1.4.30) lead to

$$g_1'^*(s) = \frac{1 - q}{Ts} \frac{1 - e^{-sT}}{1 - qe^{-sT}} \tag{1.4.32}$$

$$f_1'^*(s) = (1 - q) \frac{e^{-sT}}{1 - qe^{-sT}} \tag{1.4.33}$$

We note that a point process with skips could constitute a model for a piece of equipment designed to operate at scheduled instants of time, but which, for some reason, occasionally fails to do so.

1.4.6 Stationary point process with scheduled skips

Let the $\{t_n\}$ constitute a stationary point process. Assume that every other* occurrence (or event) is skipped; this generates a new stationary point process $\{t_n'\}$ which can be related to $\{t_n\}$. For instance,

$$P[L_1' \leq \tau] = P[L_1 \leq \tau]P[t_1' = t_1] + P[L_2 \leq \tau]P[t_1' = t_2],$$

that is

$$G_1'(\tau) = \tfrac{1}{2}\{G_1(\tau) + G_2(\tau)\} \tag{1.4.34}$$

$$g_1'^*(s) = \tfrac{1}{2}\{g_1^*(s) + g_2^*(s)\}. \tag{1.4.35}$$

Also

$$G_n'(\tau) = \tfrac{1}{2}\{G_{2n-1}(\tau) + G_{2n}(\tau)\} \tag{1.4.36}$$

$$g_n'^*(s) = \tfrac{1}{2}\{g_{2n-1}'^*(s) + g_{2n}'^*(s)\}. \tag{1.4.37}$$

*A more general case is obtained by skipping every k other occurrences (here k = 1).

Similarly, higher-order distribution can be calculated. Furthermore, if the $L_1, x_1, x_2 \ldots x_n$ are independent random variables such that all the $x_n$ are identically distributed with

$$f_1^*(s) = E[e^{-sx_n}] \tag{1.4.38}$$

then the same properties hold for $L_1'$, $x_1', \ldots x_n'$ and

$$g_1'^*(s) = \tfrac{1}{2} g_1^*(s)\{1 + f_1^*(s)\} \tag{1.4.39}$$

$$f_1'^*(s) = E[e^{-sx_n'}] = \{f_1^*(s)\}^2 \tag{1.4.40}$$

Example: The Poisson point process.

In this case, we obtain

$$g_1'^*(s) = \frac{\beta(2\beta + s)}{2(\beta + s)^2} \tag{1.4.41}$$

$$f_1'^*(s) = \left\{\frac{\beta}{\beta + s}\right\}^2 \tag{1.4.42}$$

### 1.4.7 Stationary point process with jitter

Let the $\{t_n\}$ constitute a stationary point process and suppose that $\{\epsilon_n\}$ denotes a stationary random process which is independent of the $t_n$. The set $\{t_n + \epsilon_n\}$ generates a new stationary point process $\{t_n'\}$ which can be defined in terms of the $\{t_n\}$ and $\{\epsilon_n\}$ [observe that the point process $\{t_n'\}$ is obtained by re-ordering the set $\{t_n + \epsilon_n\}$ and this may be sometimes difficult]. Let us consider an example

Example: Nearly-periodic point process

We assume that the $\{t_n\}$ constitute a periodic point process defined by equation (1. 4. 1) and that

$$-T < \epsilon_n < 0$$

with probability one.

We can write

$$P[L_1' \leq \tau] = P[0 < \alpha + \epsilon_1 \leq \tau] + P[T + \alpha + \epsilon_2 \leq \tau) \bigcap (\alpha + \epsilon_1 \leq 0)] \tag{1.4.44}$$

in view of the fact that $t_1'$ could be either $t_1 + \epsilon_1$ or $t_2 + \epsilon_2$. Similarly

$$\begin{aligned} P[L_n' \leq \tau] = {} & P[(n-1)T + \alpha + \epsilon_n \leq \tau) \bigcap (\alpha + \epsilon_1 > 0)] \\ & + P[(nT + \alpha + \epsilon_{n+1} \leq \tau) \bigcap (\alpha + \epsilon_1 \leq 0)] \end{aligned} \tag{1.4.45}$$

In an analogous way, higher-order distribution functions can be calculated.

It is of interest to observe that a stationary point process with jitter can be taken as a model for a situation when time-jitter errors are introduced in a system which is scheduled to operate at intended instants of time $t_n$.

# CHAPTER II

## Secondary Processes and Random Sampling

Summary

Following a terminology used by Takacs (ref T. 1), the random process

$$y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, \eta(t, t_n) \tag{2.0.1}$$

will be called a secondary process; here, $\eta(t, u)$ denotes a deterministic function and we assume that the $t_n$, $\alpha_n$ constitute respectively a stationary point process $\{t_n\}$ and a stationary random process $\{\alpha_n\}$ which are independent. Whenever $\eta(t, u) = \eta(t - u)$ i.e.,

$$y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, \eta(t - t_n) \tag{2.0.2}$$

we say that y(t) is a stationary secondary process. Such processes are common in modern electronic systems; a classical case is provided by the shot effect (ref D. 1., L. 1., and specially ref B. 8.). Another familiar example of a secondary process is given by the cardinal series

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{\sin \frac{\omega_0}{2}(t - t_n)}{\frac{\omega_0}{2}(t - t_n)}$$

where the $t_n$ constitute a periodic point process with period $\frac{2\pi}{\omega_0}$.

Let us consider a reciprocal situation: suppose that a continuous parameter random process x(t) has been sampled at instants $t_n$ [the $t_n$ constitute a stationary point process], thus leading to a set of samples $\{x(t_n)\}$; the problem of optimum linear interpolation consists of finding a deterministic function h(t) such that

$$\widehat{x(t)} = \sum_{n=-\infty}^{\infty} x(t_n) \, h(t - t_n) \tag{2.0.3}$$

represents the "best" recovery of x(t).

Problems of the types indicated are in general difficult to solve. By

introducing an "improper" random process, and by using some heuristic reasoning, in this chapter we suggest a possible approach to the study of (2.0.1), (2.0.2) and (2.0.3).

## 2.1 The z-process

### 2.1.1 Definition

Let us consider an enumerable sequence of rectangular pulses, with trailing edge occurring at the instant $t_n$ [the $t_n$ constitute a stationary point process]. Let d be a positive constant, and assume that the width and height of the pulse at $t_n$ are respectively d and $\frac{\alpha_n}{d}$ [the random process $\{\alpha_n\}$ is stationary]. If $t_1$ denotes the first passage time after a fixed instant t, we define a random process

$$z(t) = \begin{cases} \frac{\alpha_1}{d} & \text{if } t_1 - t \le d \\ 0 & \text{if } t_1 - t > d \end{cases} \tag{2.1.1}$$

which we call the z-process. We shall assume that $\{t_n\}$ is such that

$$\lim_{h\to 0} \frac{G_1(h)}{h} = \beta \tag{2.1.2}$$

$$\lim_{h\to 0} \frac{G_2(h)}{h} = 0 \tag{2.1.3}$$

and in view of (1.3.33), it follows that

$$E[N(\tau)] = \sum_{n=1}^{\infty} G_n(\tau) = \beta\tau . \tag{2.1.4}$$

Observe that if the pulses do not overlap (with probability one), equation (2.1.1) is equivalent to

$$z(t) = \sum_{n=-\infty}^{\infty} \alpha_n p(t - t_n) \tag{2.1.5}$$

where p(t) denotes a rectangular pulse of width d and height $\frac{1}{d}$. Moreover, as $d \to 0$, we obtain the "improper" random process

$$s(t) = \lim_{d \to 0} z(t) = \sum_{n=-\infty}^{\infty} \alpha_n \delta(t - t_n) \tag{2.1.6}$$

which consists of an infinite train of impulses (delta-functions) occurring at random instant $t_n$ with random intensity $\alpha_n$.

### 2.1.2 Statistical properties

In view of the preceding definition, the random process z(t) is stationary (in the strict sense). The following study will be limited to first and second order statistics.

First-order statistics

We have

$$E[z(t)] = E[\frac{\alpha_1}{d}] \, P[t_1 - t \leq d]$$

that is

$$\bar{z} = \frac{\alpha}{d} G_1(d) \tag{2.1.7}$$

where

$$\bar{z} = E[z(t)] \tag{2.1.8}$$

$$\alpha = E[\alpha_n] \tag{2.1.9}$$

Second-order statistics

Similarly,

$$E[z(t)^2] = E[\frac{\alpha_1^2}{d^2}] \, P[t_1 - t \leq d] \quad \text{i.e.}$$

$$R_z(0) = \frac{\rho(0)}{d^2} G_1(d) \tag{2.1.10}$$

where

$$R_z(\tau) = E[z(t)\, z(t+\tau)] \tag{2.1.11}$$

$$\rho(n) = E[\alpha_j \alpha_{j+n}] \; . \tag{2.1.12}$$

Next, we may consider $R_z(\tau)$ where without loss of generality we assume the argument $\tau$ positive. Denoting by $\{t_n\}_{n=1,2,\ldots}$ the forward point process for the instant t, and by $t_1'$ the first passage time after $t + \tau$, it is clear that $t_1'$

could be either $t_1$ or $t_2$ or $t_3 \ldots$ . Letting

$$L_1 = t_1 - t, \ L_1' = t_1' - (t + \tau) \ , \tag{2.1.13}$$

we define the joint event

$$B_n = \{(L_1 \leq d) \cap (L_1' \leq d) \cap (t_1' = t_{n+1})\} \ . \tag{2.1.14}$$

The $B_n$ being disjoint and observing that

$$E[z(t) \ z(t + \tau) / B_n] = \frac{\rho(n)}{d^2} \tag{2.1.15}$$

it follows that

$$R_z(\tau) = \frac{1}{d^2} \sum_{n=0}^{\infty} \rho(n) \ P[B_n] \ . \tag{2.1.16}$$

This correlation function can be evaluated from the process $\{L_n\}$ as follows:

i) For $\underline{\tau > d}$, interpretation of (2.1.14) leads to

$$P[B_0] = 0 \tag{2.1.17}$$

$$P[B_1] = P[(L_1 \leq d) \cap (\tau < L_2 \leq \tau + d)] \tag{2.1.18}$$

$$P[B_n] = P[(L_1 \leq d) \cap (L_n \leq \tau) \cap (\tau < L_{n+1} \leq \tau + d)], \quad n \geq 2 \tag{2.1.19}$$

ii) and for $\underline{0 < \tau \leq d}$

$$P[B_0] = P[\tau < L_1 \leq d] = G_1(d) - G_1(\tau) \tag{2.1.20}$$

$$P[B_n] = P[(L_n \leq \tau) \cap (\tau < L_{n+1} \leq \tau + d)], \quad n \geq 1. \tag{2.1.21}$$

It is interesting to observe that the z-process is continuous in the mean. In view of (2.1.21), (2.1.20), (2.1.16), (2.1.4), (2.1.10), (1.3.47), we can write

$$\left| R_z(\tau) - R_z(0) + \frac{1}{d^2} \rho(0) \ G_1(\tau) \right| \leq \frac{\rho(0) \beta \tau}{d^2}$$

and as $\tau \to 0_+$, it follows that

$$R_z(0_+) = R_z(0) \tag{2.1.22}$$

### 2.1.3 Discussion on special cases

i) Suppose that the stationary point process $\{t_n\}$ is such that, for all n

$$t_{n+1} - t_n > \ell \tag{2.1.23}$$

for some positive constant $\ell$. In other words, there exists a minimum time spacing $\ell$ between consecutive occurrences (a case commonly encountered). Let us consider some of the implications.

First we observe that

$$G_2(\ell) = P[L_2 \leq \ell] = 0$$

and, as a result of (2.1.4), it follows

$$G_1(\tau) = \beta\tau, \quad \tau \leq \ell \quad . \tag{2.1.24}$$

Then, we notice that the summation (2.1.16) is finite and can be written

$$R_z(\tau) = \frac{1}{d^2} \sum_{n=0}^{N(\tau,\ell)} \rho(n)\ P[B_n] \tag{2.1.25}$$

where

$$N(\tau, \ell) = \max\left\{k \,\middle|\, k \leq \frac{\tau}{\ell} + 1\right\}, \quad k = 1, 2, \ldots \quad . \tag{2.1.26}$$

If, in addition $d < \ell$, it can be seen from (2.1.17), ... (2.1.20)... that

$$R_z(\tau) = \begin{cases} \beta\rho(0)\,\dfrac{d - \tau}{d^2} & \text{for } \tau < d \\ \\ 0 & \text{for } d < \tau < \ell - d \end{cases} \tag{2.1.27}$$

or with a sketch

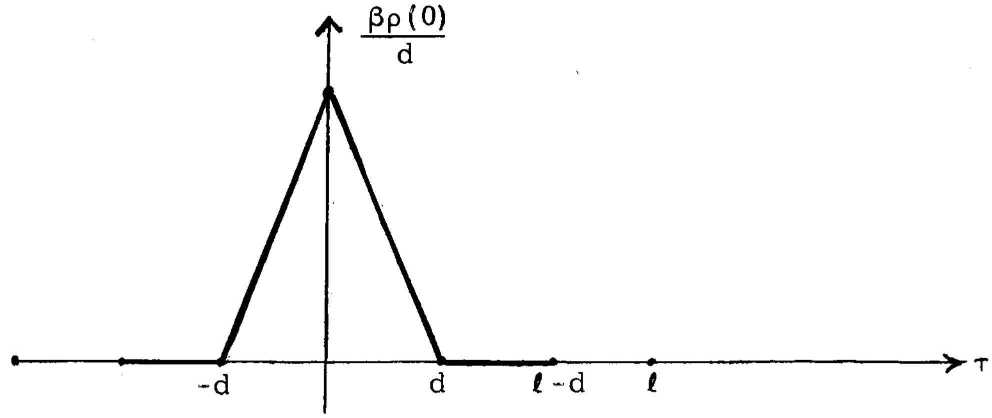


Figure 2.1. Sectional Sketch of $R_z(\tau)$.

Notice that

$$\lim_{d \to 0} R_z(\tau) = \beta\rho(0)\,\delta(\tau)\ , \quad |\tau| < \ell \ . \tag{2.1.28}$$

We observe also that the representation (2.1.5) is valid, i.e.,

$$z(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, p(t - t_n) \tag{2.1.29}$$

(p denotes a rectangular pulse of width $d < \ell$ and height $\frac{1}{d}$).

ii) Suppose that the stationary point process is such that $L_1$, $x_1, \ldots x_n$ $[x_n = L_{n+1} - L_n]$ are independent random variables and that the $x_n$ are identically distributed; we will assume that density functions exist and denote

$$f_n(\sigma) = \lim_{h \to 0} P[\sigma < \sum_{k=1}^{n} x_{j+k} \leq \sigma + h] \cdot \frac{1}{h} \quad j \in \{0, 1, \ldots\} \tag{2.1.30}$$

We shall evaluate (2.1.16) under the given assumptions.

Observing that, with $y > x$

$$P[(x < L_1 \leq x + dx) \cap (y < L_2 \leq y + dy)]$$

$$= P[x < L_1 \leq x + dx]\ P[y - x < x_1 \leq y + dy - x]$$

$$= g_1(x)dx\ f_1(y - x)dy \tag{2.1.31}$$

equations (2.1.18) and (2.1.21) (with n = 1) can be combined in the single result

$$P[B_1] = \int_0^d g_1(x)\ u(\tau - x) \int_\tau^{\tau+d} f_1(y - x)\,dy\,dx \tag{2.1.32}$$

where u(t) denotes the step (or Heaviside) function

$$u(t) = \begin{cases} 1 & , \quad t > 0 \\ 0 & , \quad t \leq 0 \end{cases} \quad . \tag{2.1.33}$$

Similarly, with $x < x + \sigma < y$, we can write

$$P[(x < L_1 \leq x + dx) \cap (\sigma + x < L_n \leq \sigma + d\sigma + x) \cap (y < L_{n+1} \leq y + dy)]$$

$$= P[x < L_1 \leq x + dx] P[\sigma < x_1 + x_{n-1} \leq \sigma + ds] P[y - x - \sigma < x_n \leq y + dy - x - \sigma]$$

$$= g_1(x)\,dx\,f_{n-1}(\sigma)\,d\sigma\,f_1(y - x - \sigma)\,dy \tag{2.1.34}$$

and it follows that eqn. (2.1.19) and (2.1.21) with $n \geq 2$ are unified by the single equation

$$P[B_n] = \int_0^d g_1(x) \int_\tau^{\tau+d} \int_0^{\tau-x} f_{n-1}(\sigma) f_1(y - x - \sigma)\,d\sigma\,dy\,dx \tag{2.1.35}$$

Finally, the correlation function $R_z(\tau)$ can be written

$$R_z(\tau) = \frac{\rho(0)}{d^2} \int_0^d g_1(x)\ u(x - \tau)\,dx$$

$$+ \frac{\rho(1)}{d^2} \int_0^d g_1(x)\ u(\tau - x) \int_\tau^{\tau+d} f_1(y - x)\,dy\,dx$$

$$+ \sum_{n=2}^{\infty} \frac{\rho(n)}{d^2} \int_0^d g_1(x) \int_\tau^{\tau+d} \int_0^{\tau-x} f_{n-1}(\sigma) f_1(y - x - \sigma)\,d\sigma\,dy\,dx, \quad \tau \geq 0. \tag{2.1.37}$$

2.1.4 <u>Example</u>

We shall assume that the $\{t_n\}$ constitute a Poisson point process (see section 1.4.2) and that

$$\alpha = 1, \qquad \rho(n) = \begin{cases} 1 + \sigma^2 & , \; n = 0 \\ 1 & , \; n \geq 1 . \end{cases} \tag{2.1.38}$$

Using (2.1.7), we obtain the average value of the z-process

$$z = \frac{1}{d}(1 - e^{-\beta d}) . \tag{2.1.39}$$

As for the correlation function $R_z(\tau)$, it will be interesting to give two different calculations.

i) For $\tau > d$, equation (2.1.16) can be written

$$\begin{aligned} R_z(\tau) &= \frac{1}{d^2} P[(L_1 \leq d) \cap (L_1' \leq d)] \\ &= \frac{1}{d^2} P[L_1 \leq d] P[L_1' \leq d] \\ &= \frac{1}{d^2}(1 - e^{-\beta d})^2 \end{aligned} \tag{2.1.40}$$

in view of the well-known fact that the Poisson point process has no memory.

For $0 \leq \tau \leq d$, we can write

$$\begin{aligned} R_z(\tau) &= \frac{1 + \sigma^2}{d^2} P[\tau < L_1 \leq d] \\ &\quad + \frac{1}{d^2} P[L_1 \leq \tau) \cap (L_1' \leq d)] \\ &= \frac{1}{d^2} [1 + \sigma^2)(e^{-\beta\tau} - e^{-\beta d}) + (1 - e^{-\beta\tau})(1 - e^{-\beta d}) \end{aligned} \tag{2.1.41}$$

ii) The same results can be obtained from (2.1.37). Observing that (for a Poisson point process)

$$\sum_{n=1}^{\infty} f_n(\sigma) = \sum_{n=1}^{\infty} g_n(\sigma) = \beta \tag{2.1.42}$$

(this may be seen from section 1.4.2), equation (2.1.37) becomes

$$R_z(\tau) = \frac{(1+\sigma^2)\beta}{d^2}\int_0^d e^{-\beta x} v(x-\tau, d)\,dx + \frac{\beta^2}{d^2}\int_0^d u(\tau - x)\int_\tau^{\tau+d} e^{-\beta y}\,dy\,dx + \frac{\beta^3}{d^2}\int_0^d u(\tau - x)\int_\tau^{\tau+d}\int_0^{\tau-x} e^{-\beta(y-\sigma)}\,d\sigma\,dy\,dx$$

$$, \quad \tau \geq 0 \qquad (2.1.43)$$

The evaluation of these elementary integrals leads to (2.1.40) and (2.1.41). A sketch of $R_z(\tau)$ is given below.

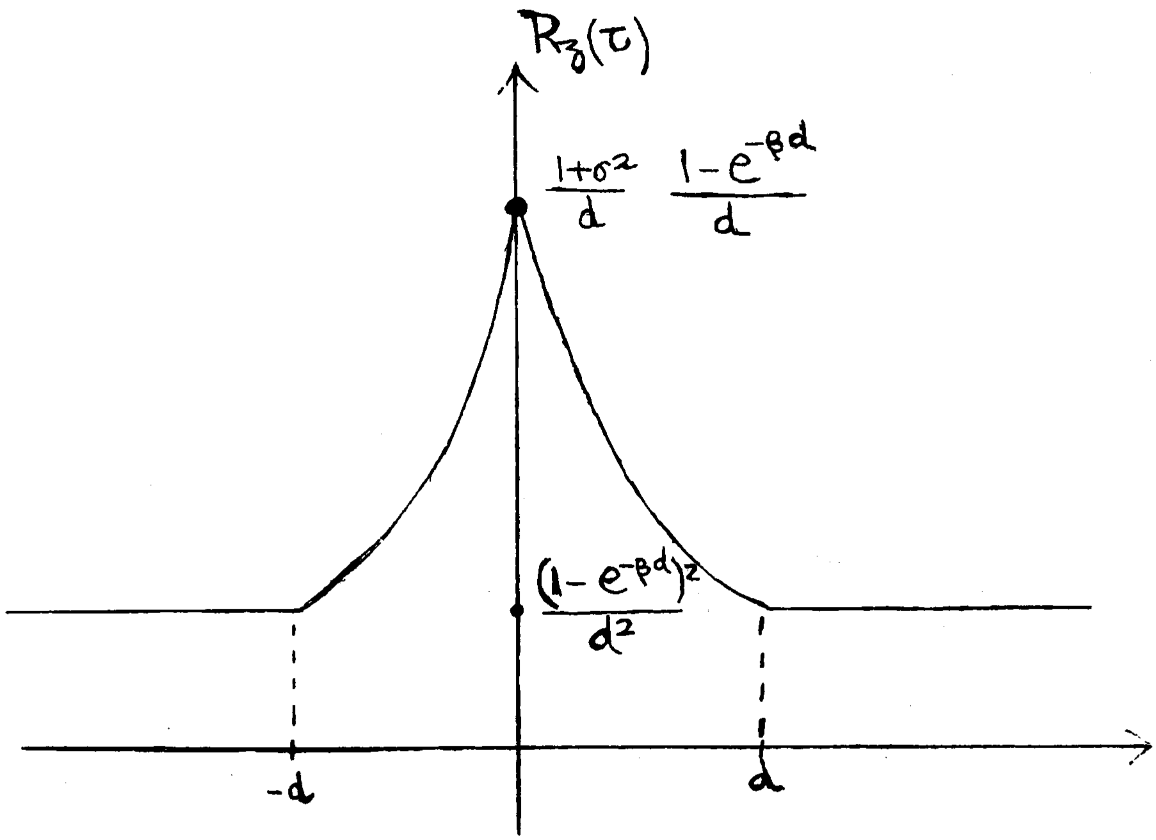


Figure 2.2. Sketch of the correlation function $R_z(\tau)$.

## 2.2 The impulse process

### 2.2.1 Definition

The impulse process s(t) is the improper random process represented as

$$s(t) = \sum_{n=-\infty}^{\infty} \alpha_n \delta(t - t_n) \tag{2.2.1}$$

Using some heuristics, we will define first and second order statistics.

### 2.2.2 Generalized statistics

Two cases will be considered.

#### 2.2.2.1 The case $\ell > 0$

We assume that the point process $\{t_n\}$ is such that (2.1.23) holds, i.e.,

$$t_{n+1} - t_n > \ell > 0 \tag{2.2.2}$$

with probability one.

In view of (2.1.29), we can write

$$s(t) = \lim_{d\to 0} z(t) \tag{2.2.3}$$

First-order statistics

Assuming $E[\lim_{d\to 0} z(t)] = \lim_{d\to 0} E[z(t)]$, equations (2.1.7), (2.1.2) lead to

$$\overline{s} = E[s(t)] = \alpha\beta \;. \tag{2.2.4}$$

Second-order statistics

Similarly, from $R_s(\tau) = E[s(t)\,s(t+\tau)] = \lim_{d\to 0} R_z(\tau)$ and because of (2.1.25), (2.1.28), (2.1.18), (2.1.19), it follows that

$$R_s(\tau) = \beta\rho(0)\,\delta(\tau) + \rho(1)\lim_{d\to 0}\frac{P[(L_1 \le d)\cap(\tau < L_2 \le \tau + d)]}{d^2}$$
$$+ \sum_{n=2}^{N(\tau,\ell)} \rho(n) \lim_{d\to 0}\frac{P[(L_1 \le d)\cap(L_n \le \tau)\cap(\tau < L_{n+1} \le \tau + d)]}{d^2}$$
$$, \quad \tau \ge 0 \tag{2.2.5}$$

thus determining the generalized correlation function $R_s(\tau)$ in terms of the statistics of the $\{L_n\}$ process.

Discussion and example

It is important to note that

$$P[(L_1 \leq d) \cap (L_n \leq \tau) \cap (\tau < L_{n+1} \leq \tau + d)] = P[(L_1 \leq d) \cap (\tau < L_{n+1} \leq \tau + d)]$$

$$, \text{ for } d < \ell \tag{2.2.6}$$

and that

$$R_s(\tau) = 0, \quad 0 < |\tau| < \ell \quad . \tag{2.2.7}$$

It is interesting to observe that if we assume the $\alpha_n$ to be uncorrelated with zero mean, this generalized correlation function becomes

$$R_s(\tau) = \beta\rho(0)\delta(\tau) \tag{2.2.8}$$

which is in accordance with the common belief that one may think of white noise as a train of uncorrelated impulses occurring randomly in time.

In order to provide an example, we shall evaluate expression (2.2.5) under the following conditions: we suppose that the instants of occurrence constitute a stationary point process with jitter, $\{t'_n\} = \{t_n + \epsilon_n\}$ [see section 1.4.7], such that

i) for all n

$$t_{n+1} - t_n > 2\ell \tag{2.2.9}$$

and such that the random variable $L_1, x_1, \ldots x_n$ associated with the $t_n$ process are independent with all the $x_n$ identically distributed according to a density function $f_1(\sigma)$

ii) the $\epsilon_n$ are pairwise independent with identical density function $f(\sigma)$ such that

$$-\ell < \epsilon_n < 0 \tag{2.2.10}$$

with probability one.

As in the example of section 1. 4. 7, we can write

$$P[(L_1' \le d)\cap(L_n' \le \tau)\cap(\tau < L_{n+1}' \le \tau + d)]$$
$$= P[(0 < L_1 + \epsilon_1 \le d)\cap(L_n + \epsilon_n \le \tau)\cap(\tau < L_{n+1} + \epsilon_{n+1} \le \tau + d)]$$
$$+ P[(L_2 + \epsilon_2 \le d)\cap(L_{n+1} + \epsilon_{n+1} \le \tau)\cap(\tau < L_{n+2} + \epsilon_{n+2} \le \tau + d)\cap(L_1 + \epsilon_1 \le 0)]$$

where the $L_n'$ are associated with the $t_n'$-process. However, it should be observed that for $d < \ell$, the preceding expression [call it J] simplifies and becomes

$$J = P[(0 < L_1 + \epsilon_1 \le d)\cap(\tau < L_{n+1} + \epsilon_{n+1} \le \tau + d)] \quad (2.\ 2.\ 11)$$

$$, d < \ell \ .$$

Thus, for $d < \ell$

$$J = P[(0 < L_1 + \epsilon_1 \le d)\cap(\tau < L_1 + x_1 + \ + x_n + \epsilon_{n+1} \le \tau + d)]$$
$$= \int_0^\infty g_1(u)\left[\int_{-u}^{d-u} f(\sigma_1)\, d\sigma_1 \int_{\tau-u}^{\tau+d-u} n(\sigma_2)\, d\sigma_2\right] du \quad (2.\ 2.\ 12)$$

where $n(\sigma)$ is the density function for $x_1 + \ldots + x_n + \epsilon_{n+1}$.

Next, we may notice that by virtue of (2. 2. 9), (2. 2. 10) and (2. 1. 24), equation (2. 2. 12) can be written

$$J = \beta \int_0^{2\ell} \left[\int_{-u}^{d-u} f(\sigma_1)\, d\sigma_1 \int_{\tau-u}^{\tau+d-u} n(\sigma_2)\, d\sigma_2\right] du \quad (2.\ 2.\ 13)$$

$$, d < \ell$$

If we assume that f and n are continuous, it then follows that

$$\lim_{d\to 0} \int_0^{2\ell} \frac{1}{d^2} [\ \ ] = \int_0^{2\ell} \lim_{d\to 0} \frac{1}{d^2} [\ \ ]^{*}$$

*This interchange of limits holds under weaker assumptions than continuity (see Hobson).

and using the theorem of the mean, we obtain

$$\lim_{d\to 0} \frac{J}{d^2} = \beta\int_0^{2\ell} f(-u)\, n(\tau - u)\,du$$

$$= \beta\int_{-\infty}^{\infty} f(-u)\, n(\tau - u)\,du$$

$$= \beta p_n(\tau) \tag{2.2.14}$$

where $p_n(\tau)$ denotes the density function for $x_1 + +x_n + \epsilon_{n+1} - \epsilon_1$. Consequently, equation (2.2.5) can be written

$$R_s(\tau) = \beta[\rho(0)\delta(\tau) + \sum_{n=1}^{N(\tau,\ell)} \rho(n)\, p_n(\tau)] \ , \quad t \geq 0 \quad . \tag{2.2.15}$$

Letting

$$\gamma(i\omega) = E[e^{-i\omega\epsilon_n}] \tag{2.2.16}$$

$$f_1^*(i\omega) = E[e^{-i\omega x_n}]$$

$$\phi_s(\omega) = \int_{-\infty}^{\infty} R_s(\tau) e^{-i\omega\tau} d\tau \tag{2.2.17}$$

and observing that

$$p_n^*(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} p_n(\tau)\,d\tau = \int_0^{\infty} e^{-i\omega\tau} p_n(\tau)\,d\tau$$
$$= |\gamma(i\omega)|^2 \{f_1^*(i\omega)\}^n ,$$

it may be seen by using Beppo-Levi's theorem (ref R. 2.) that if we assume

$$\sum_{n=1}^{\infty} |\rho(n)| < \infty \tag{2.2.18}$$

we obtain from

$$\phi_s(\omega) = \beta\rho(0) + \sum_{n=1}^{\infty} \rho(n) \int_0^{\infty} e^{-i\omega\tau} p_n(\tau) d\tau$$

$$+ \sum_{n=1}^{\infty} \rho(n) \int_0^{\infty} e^{i\omega\tau} p_n(\tau) d\tau$$

that

$$\phi_s(\omega) = \beta\rho(0)[1 - |\gamma(i\omega)|^2] + \beta|\gamma(i\omega)|^2 \sum_{n=-\infty}^{\infty} \rho(n) f_n^*(i\omega) \tag{2.2.19}$$

where

$$f_0^*(i\omega) = 1$$

and where for $n > 0$

$$f_{-n}^*(i\omega) = f_n^*(-i\omega) = \{f_1^*(-i\omega)\}^n .$$

Illustration: Nearly-periodic point process with skips.

We have for the periodic process with skips, from (1.4.33),

$$f_1^*(i\omega) = \frac{(1-q)e^{-i\omega T}}{1 - q e^{-i\omega T}}$$

and from (1.4.21)

$$\beta = \frac{1-q}{T} ,$$

and we shall assume that

$$\rho(0) = 1 + \sigma^2 \tag{2.2.20}$$

$$\rho(n) = \rho^{|n|}, \quad |\rho| < 1, \quad |n| \geq 1.$$

The evaluation of the infinite series (2.2.19) leads to

$$\phi_s(\omega) = \frac{1-q}{T}[1+\sigma^2] + \frac{1-q}{T}|\gamma(i\omega)|^2 \sum_{n=1}^{\infty} \{\rho \frac{(1-q)e^{-i\omega T}}{1 - q e^{-i\omega T}}\}^n$$

$$+ \frac{1-q}{T}|\gamma(i\omega)|^2 \sum_{n=1}^{\infty} \{\rho \frac{(1-q)e^{i\omega T}}{1 - q e^{i\omega T}}\}^n$$

and

$$\phi_s(\omega) = \frac{1-q}{T}\left[1+\sigma^2 - \frac{\rho\,|\gamma(i\omega)|^2(1-q)}{\mu}\right] + \rho\,\frac{(1-q)^2}{T}\,\frac{|\gamma(i\omega)|^2}{\mu}\,\frac{1-\mu^2}{1-2\mu\cos\omega T+\mu^2}$$

where

$$\mu = \rho + q - \rho q \ . \tag{2. 2. 21}$$

Two limiting cases for the preceding expression should be considered: in view of the following properties of a Poisson kernel (ref W. 2.)

$$\lim_{\mu\to+1} \frac{1-\mu^2}{1-2\mu\cos\omega T+\mu^2} = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0)$$

$$\lim_{\mu\to-1} \frac{1-\mu^2}{1-2\mu\cos\omega T+\mu^2} = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta[\omega - (n+\tfrac{1}{2})\omega_0] \ , \qquad \omega_0 = \frac{2\pi}{T}$$

and observing that

$$\mu = 1, \quad 0 \le q < 1 \Longleftrightarrow \rho = 1$$

$$\mu = -1 \Longleftrightarrow \rho = -1, \quad q = 0 \ ,$$

it follows that in a generalized function sense, we can write:

when

$$\rho = 1, \quad q < 1 \tag{2. 2. 22}$$

then

$$\phi_s(\omega) = \frac{1-q}{T}\left[1+\sigma^2-(1-q)|\gamma(i\omega)|^2\right]+(1-q)^2\,|\gamma(i\omega)|^2\,\frac{2\pi}{T^2}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0) \tag{2. 2. 23}$$

and when

$$\rho = -1, \quad q = 0 \tag{2. 2. 24}$$

then

$$\phi_s(\omega) = \frac{1}{T}\left[1-|\gamma(i\omega)|^2+\sigma^2\right]+|\gamma(i\omega)|^2\,\frac{2\pi}{T^2}\sum_{n=-\infty}^{\infty}\delta[\omega-(n+\tfrac{1}{2})\omega_0] \tag{2. 2. 25}$$

The case (2. 2. 22) with $q = 0$ (no skips) is well-known. The case (2. 2. 24) with $\sigma^2 = 0$ corresponds to an alternating impulse process

$$s(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t - t_n)$$

where the $t_n$ constitute a nearly-periodic point process.

2.2.2.2 The case $\ell = 0$

Assumption (2.1.3) indicates that the probability of having two or more occurrences in a small interval is negligible; as a result, the case $\ell = 0$ could be considered as the limiting situation of section (2.2.2.1) when $\ell \to 0$. In view of (2.2.4), (2.2.5), (2.1.26), it follows that

$$\overline{s} = E[s(t)] = \alpha\beta \qquad (2.2.27)$$

and

$$R_s(\tau) = \beta\rho(0)\delta(\tau) + \rho(1) \lim_{d\to 0} \frac{P[(L_1 \le d)\cap(\tau < L_2 \le \tau + d)]}{d^2}$$

$$+\sum_{n=2}^{\infty} \rho(n) \lim_{d\to 0} \frac{P[(L_1 \le d)\cap(L_n \le \tau)\cap(\tau < L_{n+1} \le \tau + d)]}{d^2} \qquad (2.2.28)$$

$$, \quad \tau \ge 0$$

Discussion and example

We have found it difficult to give a rigorous justification for what precedes.

Let us evaluate (2.2.28) under the assumption that $L_1, x_1, \ldots x_n$ are independent random variables such that all the $x_n$ are identically distributed according to a continuous density function $f_1(\sigma)$. It may be seen from equations (2.1.32), (2.1.35) that (2.2.28) leads to

$$R_s(\tau) = \beta[\rho(0)\delta(\tau) + \sum_{n=1}^{\infty} \rho(n) f_n(\tau)] \quad , \quad \tau \ge 0 \qquad (2.2.29)$$

with

$$f_n^*(s) = \{f_1^*(s)\}^n .$$

Notice that in view of

$$g_n^*(s) = g_1^*(s)\{f_1^*(s)\}^{n-1}$$

$$\sum_{n=1} g_n^*(s) = \frac{\beta}{s}, \quad \mathrm{Re}(s) > 0 \qquad *$$

we have

$$g_1^*(s) = \beta \frac{1 - f_1^*(s)}{s} \tag{2.2.30}$$

thus identifying $f_n(\tau)$ as the derivative of the distribution function $F_n(\tau)$ which appears in equations (1.3.36), (1.3.37). Also, we should note that if the point process $\{t_n\}$ satisfies

$$E[N(\tau)^2] < \infty \tag{2.2.31}$$

it then follows from (1.3.74) that

$$\left|\sum_{n=1}^{\infty} \rho(n) f_n(\tau)\right| \leq \rho(0) \sum_{n=1}^{\infty} f_n(\tau) < \infty \quad \text{a. e.} \tag{2.2.32}$$

If we assume on the other hand that condition (2.2.18) holds, we obtain

$$\phi_s(\omega) = \beta \sum_{n=-\infty}^{\infty} \rho(n) f_n^*(i\omega) \tag{2.2.33}$$

where

$$f_0^*(i\omega) = 1$$

$$f_{-n}^*(i\omega) = f_n^*(-i\omega) = \{f_1^*(-i\omega)\}^n \; .$$

<u>Illustration: Poisson point process</u>

We consider various choices for the $\rho(n)$:

i) if we take

*We recall that from theorem 4, $\sum_{n=1}^{\infty} G_n(\tau) = \beta\tau$ and $\sum_{n=1}^{\infty} g_n(x) = \beta$ a. e.

$$\alpha = 1, \quad \rho(0) = 1 + \sigma^2$$
$$\rho(n) = 1 \quad (n \geq 1) \tag{2.2.34}$$

we obtain from (2. 2. 27) that

$$\overline{s} = \beta$$

and from (2. 1. 42) and (2. 2. 29)

$$R_s(\tau) = \beta(1 + \sigma^2)\delta(\tau) + \beta^2 \tag{2.2.35}$$

$$\phi_s(\omega) = \beta(1 + \sigma^2) + 2\pi\beta^2\delta(\omega) \ . \tag{2.2.36}$$

ii) if

$$\alpha = 0 \ , \quad \rho(n) = (-1)^n \tag{2.2.37}$$

we have

$$\overline{s} = 0 \tag{2.2.38}$$

and

$$R_s(\tau) = \beta\delta(\tau) - \beta^2 e^{-2\beta\tau} \tag{2.2.39}$$
$$, \tau \geq 0$$

$$\phi_s(\omega) = \beta \frac{\omega^2}{\omega^2 + 4\beta^2} \ . \tag{2.2.40}$$

It is interesting to observe that the case (2. 2. 37) occurs when we have an alternating impulse process described as

$$s(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t - t_n) \tag{2.2.41}$$

where the $\{t_n\}$ constitute a Poisson point process.

iii) if we choose

$$\rho(n) = \rho^{|n|}, \quad |\rho| < 1 \tag{2.2.42}$$

it follows that condition (2. 2. 18) is satisfied and consequently from

(2. 2. 33), we obtain

$$\phi_s(\omega) = \beta \frac{\omega^2 + \beta^2(1 - \rho^2)}{\beta^2(1 - \rho)^2 + \omega^2} \quad . \tag{2. 2. 43}$$

## 2. 3 Applications to secondary process

Let us consider a secondary process

$$y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \eta(t, t_n) \tag{2. 3. 1}$$

which we write

$$y(t) = \int_{-\infty}^{\infty} \eta(t, \sigma) s(\sigma) d\sigma \tag{2. 3. 2}$$

where

$$s(t) = \sum_{n=-\infty}^{\infty} \alpha_n \delta(t - t_n) \tag{2. 3. 3}$$

and $\{t_n\}$ is a stationary point process. In other words, we are viewing a secondary process y(t) as the output of a linear time-varying system (acting as a shaping filter) having as an input the impulse process s(t).

Let us assume that equation (2. 3. 2) may be handled as if s(t) is a perfectly "good" process. Under such conditions, we obtain from (2. 2. 27)

$$E[y(t)] = \alpha\beta \int_{-\infty}^{\infty} \eta(t, \sigma) d\sigma \tag{2. 3. 4}$$

$$E[y(t)y(t + \tau)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(t, \sigma_1) \, \eta(t + \tau, \sigma_2) R_s(\sigma_1 - \sigma_2) d\sigma_1 d\sigma_2 \tag{2. 3. 5}$$

assuming convergence of these integrals [$R_s$ is defined by equation (2. 2. 5) or (2. 2. 28)].

It is interesting to consider the case when y(t) is a stationary secondary process, i. e.;

$$y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \eta(t - t_n) \quad . \tag{2. 3. 6}$$

Here, equations (2. 3. 4), (2. 3. 5) lead to

$$\overline{y} = E[y(t)] = \alpha\beta \int_{-\infty}^{\infty} \eta(\sigma)\, d\sigma \tag{2. 3. 7}$$

$$\begin{aligned} R_y(\tau) &= E[y(t + \tau)y(t)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta(\sigma_1)\eta(\sigma_2) R_s(\tau + \sigma_1 - \sigma_2)\, d\sigma_1\, d\sigma_2 \end{aligned} \tag{2. 3. 8}$$

There are cases where the classical Parseval relation (in $L_2$) can be extended to hold in a more generalized Fourier theory (ref L. 4.). Under such conditions, equation (2. 3. 8) would lead to the familiar relation

$$\phi_y(\omega) = |\eta^*(i\omega)|^2 \phi_s(\omega) \ , \tag{2. 3. 9}$$

where $\eta^*(i\omega)$ is the Fourier transform of $\eta(t)$, thus giving the spectral density associated with y(t) in terms of the generalized Fourier transforms of $\eta(t)$ and $R_s(t)$.

Examples

1) The coin-tossing process

This process is well-known (see for instance ref L. 1, page 128, ...). It is defined as

$$y(t) = \alpha_n \ , \quad t_n < t \leq t_{n+1} \quad ; \tag{2. 3.10}$$

where $\{t_n\}$ is a periodic point process with period one, and the $\alpha_n$ are independent random variables taking on the values zero or one with respective probabilities q and 1 - q.

It is interesting to observe that y(t) can be interpreted as a secondary process in two equivalent ways:

$$y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \eta(t - t_n)$$

or

$$y(t) = \sum_{n=-\infty}^{\infty} \eta(t - t'_n)$$

where $\eta$ is a rectangular pulse of unit height and unit width and where $\{t'_n\}$ constitutes a periodic point process with skips. Equations (2.3.13), (2.3.7) (2.3.9), (2.2.23 with $\sigma^2 = 0$ and $\gamma(i\omega) = 1$) lead to

$$E[y(t)] = 1 - q \tag{2.3.14}$$

$$\phi_y(\omega) = q(1 - q) \frac{\sin^2 \frac{\omega}{2}}{(\frac{\omega}{2})^2} + 2\pi(1 - q)^2 \delta(\omega) \tag{2.3.15}$$

or

$$R_y(\tau) = \begin{cases} -q(1 - q)|\tau| + (1 - q), & 0 \le |\tau| \le 1 \\ (1 - q)^2, & |\tau| > 1 \end{cases} \tag{2.3.16}$$

2) <u>The random telegraph wave</u>

This process is usually defined as

$$y(t) = (-1)^n, \quad t_n < t \le t_{n+1}$$

where the set $\{t_n\}$ constitutes a Poisson point process. y(t) could be considered as the output of an integrator with input

$$s(t) = \sum_{n=-\infty}^{\infty} 2(-1)^n \delta(t - t_n).$$

Using (2. 3. 7), (2. 3. 9), (2. 2. 37), (2. 2. 40) and in view of

$$|\eta^*(i\omega)|^2 = \frac{1}{\omega^2} \quad ,$$

we obtain

$$E[\mathbf{y}(t)] = 0$$

$$\phi_y(\omega) = \frac{4\beta}{\omega^2 + 4\beta^2}$$

or

$$R_y(\tau) = e^{-2\beta|\tau|}$$

which are well-known results.

3) Poisson emission of pulses

We consider

$$y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \eta(t, t_n) \tag{2. 3.17}$$

where $\{t_n\}$ is a Poisson point process and where the $\alpha_n$ are assumed to satisfy (2. 2. 34)

Using (2. 3. 4), (2. 3. 5), (2. 2. 35), we obtain immediately

$$E[y(t)] = \beta \int_{-\infty}^{\infty} \eta(t, \sigma) d\sigma \tag{2. 3.18}$$

$$E[y(t)y(t + \tau)] = \beta(1 + \sigma^2) \int_{-\infty}^{\infty} \eta(t, \sigma)\eta(t + \tau, \sigma) d\sigma$$

$$+ E[y(t)]E[y(t + \tau)] \tag{2. 3.19}$$

results which agree with ref L. l., pages 149, 151.

In particular, if y(t) is a stationary secondary process, we obtain

$$\overline{y} = E[y(t)] = \beta \int_{-\infty}^{\infty} \eta(\sigma) d\sigma \tag{2. 3. 20}$$

and

$$\phi_y(\omega) = \beta(1 + \sigma^2) |\eta^*(i\omega)|^2 + 2\pi y^2 \delta(\omega) \tag{2.3.21}$$

The preceding equations are useful for studying the current y(t) at the anode in a pentode; if $\eta(t - \sigma)$ denotes the current pulse produced at the anode by an electron emitted from the cathode at the instant $\sigma$ and if q is the probability of interception by the grid , we can write

$$y(t) = \sum_{n=-\infty}^{t_n \leq t} \eta(t - t_n') \tag{2.3.22}$$

where $\{t_n'\}$ is a Poisson point process with skips. Knowing that $\{t_n'\}$ is a Poisson point process with parameter $\beta(1 - q)$, it follows from (2.3.21), (2.3.22) that

$$\overline{y} = (1 - q)\beta \int_0^\infty \eta(\sigma)\, d\sigma \tag{2.3.23}$$

$$\phi_y(\omega) = \beta(1 - q) |\eta^*(i\omega)|^2 + 2\pi y^2 \delta(\omega) \quad . \tag{2.3.24}$$

For a diode, we take q = 0 and in this case the preceding equations bear the name of Campbell's theorem.

Illustrations

i) The infinite "Poisson" cardinal series

We define this process as

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{\sin \frac{\omega_0}{2}(t - t_n)}{\frac{\omega_0}{2}(t - t_n)} \tag{2.3.25}$$

where $\{t_n\}$ is a Poisson process.

From equations (2. 3. 20), (2. 3. 21, $\sigma^2 = 0$) and in view of

$$|\eta^*(i\omega)|^2 = \begin{cases} T^2 & , \ 0 \le |\omega| \le \frac{\omega_0}{2} \\ 0 & , \ |\omega| > \frac{\omega_0}{2} \end{cases}$$

$$, \ T = \frac{2\pi}{\omega_0} \qquad (2.3.26)$$

we obtain

$$E[y(t)] = \beta T \qquad (2.3.27)$$

$$R_y(\tau) = \beta^2 T^2 + \beta T \frac{\sin \frac{\omega_0}{2}\tau}{\frac{\omega_0}{2}\tau} \qquad (2.3.28)$$

ii) The semi-infinite "Poisson" cardinal series

This process will be defined as

$$y(t) = \sum_{n=-\infty}^{t_n \le t} \frac{\sin \frac{\omega_0}{2}(t - t_n)}{\frac{\omega_0}{2}(t - t_n)} \quad . \qquad (2.3.29)$$

Since

$$\eta(t) = \begin{cases} \dfrac{\sin \frac{\omega_0}{2} t}{\frac{\omega_0}{2} t} & , \ t > 0 \\ 0 & , \ t \le 0 \end{cases} \qquad (2.3.30)$$

or

$$|\eta^*(i\omega)|^2 = \frac{4}{\omega_0^2} \left| \mathrm{Arc\,tg} \frac{\omega_0}{2i\omega} \right|^2 \quad , \qquad (2.3.31)$$

we obtain this time

$$E[y(t)] = \tfrac{1}{2} \beta T \qquad (2.3.32)$$

$$\phi_y(\omega) = 2\pi \frac{\beta^2 T^2}{4} \delta(\omega) + \beta \frac{4}{\omega_0^2} \left| \operatorname{Arc\,tg} \frac{\omega_0}{2i\omega} \right|^2 \tag{2.3.33}$$

$$R_y(\tau) = \tfrac{1}{4}\beta^2 T^2 + \beta \frac{4}{\omega_0^2} \int_0^\infty \frac{\sin(\frac{\omega_0}{2}\sigma) \sin \frac{\omega_0}{2}(|\tau| + \sigma)}{\sigma(|\tau| + \sigma)} d\sigma$$

$$, \; T = \frac{2\pi}{\omega_0} \tag{2.3.34}$$

and observe that

$$R_y(0) = E[y(t)^2] = \tfrac{1}{4}\beta^2 T^2 + \tfrac{1}{2}\beta T \; . \tag{2.3.35}$$

iii) <u>The infinite train of rectangular pulses</u>

We consider

$$y(t) = \sum_{n=-\infty}^{\infty} \alpha_n \eta(t - t_n)$$

where the $\alpha_n$ satisfy (2.2.34) and where $\eta(t)$ is a rectangular pulse of width d and height $\frac{1}{d}$. Therefore

$$|\eta^*(i\omega)|^2 = \frac{\sin^2 \frac{\omega d}{2}}{(\frac{\omega d}{2})^2} \tag{2.3.36}$$

As a result, we obtain

$$E[y(t)] = \beta \tag{2.3.37}$$

$$\phi_y(\omega) = 2\pi\beta^2 \delta(\omega) + \beta(1 + \sigma^2) \frac{\sin^2 \frac{\omega d}{2}}{(\frac{\omega d}{2})^2} \tag{2.3.38}$$

and

$$R_y(\tau) = \begin{cases} -\frac{\beta}{d^2}(1 + \sigma^2)|\tau| + \beta^2 + \frac{\beta}{d}(1 + \sigma^2) & , \; 0 \le |\tau| \le d \\ \\ \beta^2 & , \; |\tau| > d \end{cases} \tag{2.3.39}$$

It is interesting to compare these results with equation (2.1.39), (2.1.40), (2.1.41). [See also Fig. 2.3.].

## 2.4 Applications to random sampling

### 2.4.1 Introduction

The problem of linear interpolation is well-known and has been briefly defined in the summary for this chapter. However, in this definition, we did not take into account the errors introduced by the sampling mechanism: errors in amplitude and errors in timing (assuming zero-width sampling). As a result, the problem defined by equation (2.0.3) will be simply restated in terms of

$$\widehat{x(t)} = \sum_{n=-\infty}^{\infty} (1 + a_n)x(t_n + e_n)h(t - t_n) , \qquad (2.4.1)$$

where $e_n$ denotes the time-jitter error and $a_n$ represents a scaling error in amplitude. Equation (2.4.1) can be written

$$\widehat{x(t)} = \int_{-\infty}^{\infty} h(t - \sigma)y(\sigma)\,d\sigma \qquad (2.4.2)$$

where

$$y(t) = x_1(t)\, s(t) \qquad (2.4.3)$$

with

$$s(t) = \sum_{n=-\infty}^{\infty} (1 + a_n)\, \delta(t - t_n) \qquad (2.4.4)$$

$$x_1(t) = x[t + e(t)] \qquad (2.4.5)$$

and

$$e(t_n) = e_n \quad . \qquad (2.4.6)$$

Our aim is to find the interpolation function h(t) which will allow the "best" recovery of x(t) [$\widehat{x(t)}$ is often called the best estimate of x(t)]. By choosing as index of performance the minimum mean-square error criterion, we shall solve this problem by employing a heuristic extension of Wiener's filtering theory.

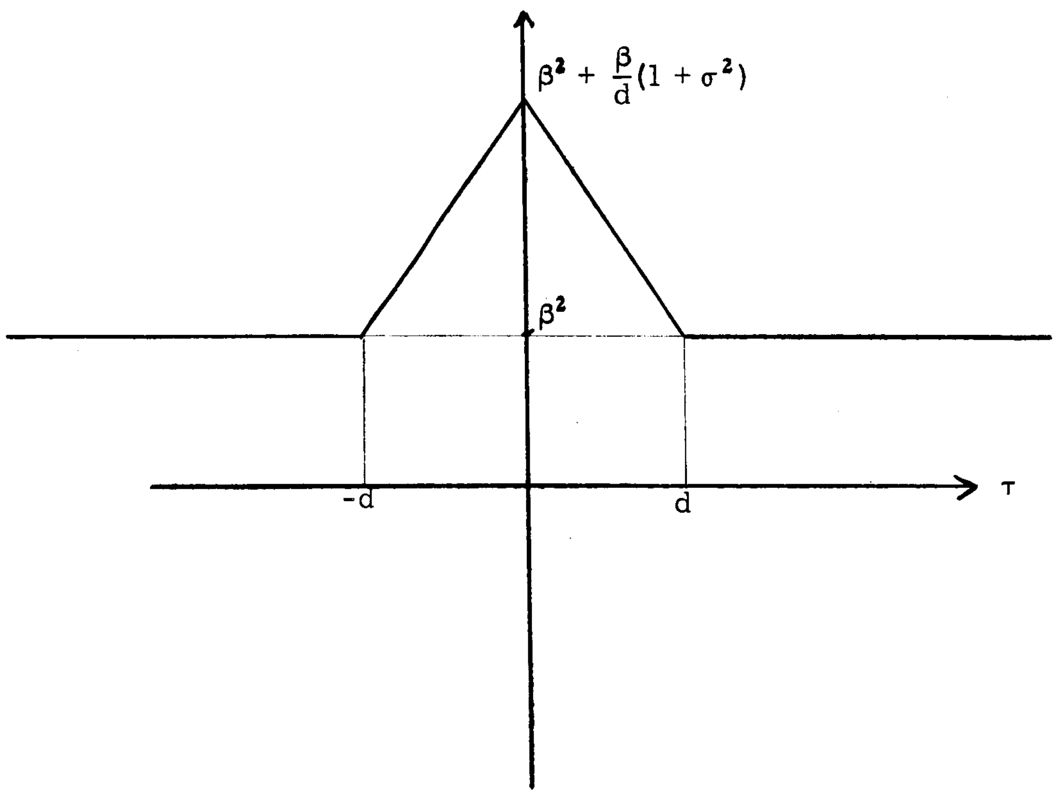


Figure 2. 3. Correlation function for the infinite train of rectangular pulses.

### 2.4.2 Brief review on Wiener's theory for filtering.

The concept of this theory is familiar and we shall briefly summarize some classical results by considering a simplified case (for the details on the theory, we refer to W.3, B.7, D.4, L.1).

Assuming that

$$R_x(\tau) = E[x(t + \tau)\ x(t)] \tag{2.4.7}$$

$$R_y(\tau) = E[y(t + \tau)\ y(t)] \tag{2.4.8}$$

$$R_{xy}(\tau) = E[x(t + \tau)\ y(t)] \tag{2.4.9}$$

are continuous functions and that each has a Fourier transform [respectively $\phi_x(\omega)$, $\phi_y(\omega)$, $\phi_{xy}(\omega)$], we want to find the function h(t) such as to minimize

$$\epsilon^2 = E[\,|x(t) - \widehat{x(t)}\,|^2\,] \tag{2.4.10}$$

where

$$\widehat{x(t)} = \int_{-\infty}^{\infty} h(t - \sigma)\ y(\sigma)\, d\sigma \tag{2.4.11}$$

This problem leads to

i) A non-realizable solution h(t) obtained from

$$H(\omega) = \frac{\phi_{xy}(\omega)}{\phi_y(\omega)} \tag{2.4.12}$$

where $H(\omega)$ denotes the Fourier transform of h(t) [possibly in an extended sense].

ii) A realizable solution h(t) given by the Wiener-Hopf integral equation

$$R_{xy}(\tau) = \int_0^{\infty} h(\sigma)\, R_y(\tau - \sigma)\, d\sigma\ , \qquad \tau \geq 0\ ; \tag{2.4.13}$$

using complex variables techniques, it can be shown that if

$$\int_{-\infty}^{\infty} \frac{\operatorname{Log}\phi_y(\omega)}{1 + \omega^2}\, d\omega \qquad -\infty \tag{2.4.14}$$

(We may note that $\phi_y(\omega) = e^{-\omega^2}$ violates condition (2. 4. 14).)

then

$$H(\omega) = \frac{1}{2\pi G_1(\omega)} \int_0^{\infty} e^{-i\omega t} \left\{ \int_{-\infty}^{\infty} \frac{\phi_{xy}(u)}{G_2(u)} e^{itu} du \right\} dt \tag{2. 4. 15}$$

where

$$G_2(u + iv) = e^{\lambda(u+iv)} = P(u, v) + iQ(u,v) \tag{2. 4. 16}$$

$$G_1(u + iv) = P(u, -v) - iQ(u, -v) \tag{2. 4. 17}$$

with

$$\lambda(z) = \frac{i}{\pi} \int_0^{\infty} \frac{z \operatorname{Log} \phi_y(s)}{z^2 - s^2} ds \; ; \tag{2. 4. 18}$$

$G_2$ (or $G_1$) is the solution* of the so-called factorization problem

$$\phi_y(\omega) = G_2(\omega)\, G_1(\omega) = G_2(\omega)\, \overline{G_2(\omega)}, \text{ for all real } \omega \; ; \tag{2. 4. 19}$$

here, not only $G_2(u + iv)$ is analytic and bounded in the upper half plane, but also $\frac{1}{G_2(u + iv)}$ is analytic in the upper half plane; similar properties hold for $G_1(u + iv)$ in the lower half plane [there are situations, e. g., the rational case, where $G_2$ and $G_1$ may be simply found by inspection of the zeros and poles of $\phi_y(\omega)$].

In any case, the minimum mean-square error is given as

$$\epsilon^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\phi_x(\omega) - |H(\omega)|^2 \phi_y(\omega)] d\omega \quad . \tag{2. 4. 20}$$

It is clear that the preceding filtering problem resembles our interpolation problem, except for the fact that the process y(t) of equation (2. 4. 2) is

---

*$G_2$ is a. e. unique up to a complex constant of modulus unity.

an improper random process. However, there are cases where the generalized function $R_y(\tau)$ * exists ** and has a generalized Fourier transform $\phi_y(\omega)$ which is non-negative; furthermore, under some conditions **, the classical Parseval relation (in $L_2$) may be extended to hold in a generalized Fourier theory. In view of these facts, we shall heuristically extend the preceding results and give some examples.

### 2.4.3 Optimum interpolation

Let us consider equations (2.4.2), ... and, for simplicity, assume that:

i) $x(t)$ is weakly stationary, continuous in the mean and has a spectral density $\phi_x(\omega)$.

ii) the "time-jitter process" $e(t)$ is such that, for $u \neq v$, $e(u)$ and $e(v)$ are independent random variables identically distributed with characteristic function

$$C(i\omega) = E[e^{-i\omega e(u)}] \quad . \tag{2.4.21}$$

iii) The processes $\{t_n\}$, $\{a_n\}$, $e(t)$, $x(t)$ are mutually independent and

$$E[a_n] = 0$$

$$E[a_n a_m] = \begin{cases} \sigma^2 & , \; n = m \\ 0 & , \; n \neq m \end{cases} \tag{2.4.22}$$

Denoting

$$R_{x_1}(\tau) = E[x_1(t+\tau)\, x_1(t)] \tag{2.4.23}$$

$$R_{xx_1}(\tau) = E[x(t+\tau)\, x_1(t)] \tag{2.4.24}$$

---

* $R_y(\tau)$ denotes the extended correlation function associated with $y(t)$.

** For instance, see ref L. 4, L. 2, G. 1, E2.

we obtain

$$R_y(\tau) = R_{x_1}(\tau) R_s(\tau) \qquad (2.4.25)$$

$$R_{xy}(\tau) = R_{xx_1}(\tau) \quad \beta \ . \qquad (2.4.26)$$

Furthermore, from

$$R_{x_1}(\tau) = E\{R_x[\tau + e(t+\tau) - e(t)]\} \qquad (2.4.27)$$

and in view of

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(u)\, e^{iu\tau}\, du \qquad (2.4.28)$$

we can write

$$R_{x_1}(0) = R_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(u)\, du \qquad (2.4.29)$$

$$R_{x_1}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(u) |C(iu)|^2 e^{iu\tau}\, du$$

$$, \ \tau \neq 0 \qquad (2.4.30)$$

[Observe that, in general, $x_1(t)$ is not continuous in the mean.]

Let

$$r(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(u) |C(iu)|^2 e^{iu\tau}\, du \qquad (2.4.31)$$

and

$$a^2 = R_x(0) - r(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(u) [1 - |C(iu)|^2] du \ . \qquad (2.4.32)$$

Then in view of the delta-function which appears in (2.2.28), equation (2.4.25) can be written

$$R_y(\tau) = \beta(1+\sigma^2) a^2 \delta(\tau) + r(\tau) \ R_s(\tau) \qquad (2.4.33)$$

or by using generalized Fourier transforms

$$\phi_y(\omega) = \beta(1+\sigma^2) a^2 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(u) |C(iu)|^2 \phi_s(\omega - u)\, du \ . \qquad (2.4.34)$$

Similarly, we obtain

$$R_{xy}(\tau) = \beta \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(u) C(iu) e^{iu\tau}\, du \tag{2.4.35}$$

or

$$\phi_{xy}(\omega) = \beta \phi_x(\omega) C(i\omega) \quad . \tag{2.4.36}$$

Next, we shall illustrate with some examples.

2.4.3.1 Nearly-periodic sampling with skips, time-jitter and amplitude errors

Referring to (2.2.23), the preceding equations give

$$\phi_y(\omega) = \left(\frac{1-q}{T}\right)^2 \sum_{n=-\infty}^{\infty} \phi_x(\omega - n\omega_0) |\gamma(in\omega_0) C(i\omega - in\omega_0)|^2$$

$$+ \frac{1-q}{T}(1+\sigma^2) R_x(0) - \frac{(1-q)^2}{2\pi T} \int_{-\infty}^{\infty} \phi_x(u) |C(iu)\gamma(i\omega - iu)|^2 du$$

$$, \quad \omega_0 = \frac{2\pi}{T} \tag{2.4.37}$$

and

$$\phi_{xy}(\omega) = \frac{1-q}{T} \phi_x(\omega) C(i\omega) \ . \tag{2.4.38}$$

It is perhaps interesting to observe that in view of

$$\sum_{n=-\infty}^{\infty} \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \phi_x(\omega - n\omega_0)\, d\omega = \int_{-\infty}^{\infty} \phi_x(\omega)\, d\omega < \infty$$

it follows from Beppo-Levi's theorem that

$$\phi_y(\omega) < \infty \quad \text{a.e.}$$

and that furthermore $\phi_y(\omega)$ is locally summable. Also notice that for those values of $\omega$ where the summation $\sum_{n=-\infty}^{\infty} \phi_x(\omega - n\omega_0)$ vanishes, $\phi_x(\omega)$ must also vanish.

Case I: Non-realizable interpolator

From (2. 4. 12), we obtain

$$H(\omega) = \frac{T}{1-q} \frac{\phi_x(\omega)C(i\omega)}{\sum_{n=-\infty}^{\infty} \phi_x(\omega - n\omega_0)|\gamma(in\omega_0)C(i\omega - in\omega_0)|^2 + \frac{(1+\sigma^2)TR_x(0)}{1-q} - \frac{T}{2\pi}\int_{-\infty}^{\infty} \phi_x(u)|C(iu)\gamma(i\omega - iu)|^2 du} \tag{2. 4. 40}$$

If we assume that the sampled process x(t) is band-limited such that

$$\phi_x(\omega) = 0, \quad |\omega| \geq \frac{\omega_0}{2} \tag{2. 4. 41}$$

the preceding expression leads to

$$H(\omega) = \frac{T}{1-q} \frac{\phi_x(\omega)C(i\omega)}{\phi_x(\omega)|C(i\omega)|^2 + \frac{T}{1-q}(1+\sigma^2)R_x(0) - \frac{T}{2\pi}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \phi_x(u)|C(iu)\gamma(i\omega - iu)|^2 du} \tag{2. 4. 42}$$

with an associated mean-square error

$$\epsilon^2 = \frac{1}{2\pi}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \frac{H(\omega)}{C(i\omega)} \left[(1+\sigma^2)R_x(0) - \frac{1-q}{2\pi}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \phi_x(u)|C(iu)\gamma(i\omega - iu)|^2 du\right] d\omega \tag{2. 4. 43}$$

By letting q = 0 (no skips), $\sigma^2 = 0$ (no errors in amplitude), the preceding results agree with those in references B. 2 and B. 9.

Let us apply (2. 4. 42) to a specific example: assume that the sampled process x(t) is wide-sense Markov, i. e.,

$$R_x(\tau) = e^{-a|\tau|}, \quad a > 0 \tag{2.4.44}$$

and that

$$q = 0, \; C(i\omega) = 1, \quad \gamma(i\omega) = 1, \; \sigma^2 = 0 \tag{2.4.45}$$

(in other words, we have an ideal periodic sampling). In this case, we have

$$H(\omega) = T \frac{\phi_x(\omega)}{\sum_{n=-\infty}^{\infty} \phi_x(\omega - n\omega_0)} \tag{2.4.46}$$

and in view of (Poisson summation, ref L. 4., page 70)

$$\sum_{n=-\infty}^{\infty} \phi_x(\omega - n\omega_0) = T \sum_{n=-\infty}^{\infty} R_x(nT) \, e^{-in\omega T} \tag{2.4.47}$$

we obtain

$$H(\omega) = \frac{2a}{1 - e^{-2aT}} \; \frac{1 - 2e^{-aT} \cos \omega T + e^{-2aT}}{a^2 + \omega^2} \; . \tag{2.4.48}$$

Then the interpolation function is

$$h(t) = \begin{cases} e^{-a|t|} & , \; 0 < |t| \le T \\ 0 & , \; \text{otherwise} \end{cases} \; ; \tag{2.4.49}$$

it is clear that the interpolation function uses only two samples: the most recent ones from the right and from the left [this result is not too surprising in view of the Markovian property of $x(t)$].

<u>Case II: Realizable interpolator</u>

Let us suppose that

$$R_x(\tau) = e^{-a|\tau|}, \quad a > 0 \tag{2.4.50}$$

and that

$$C(i\omega) = 1, \; \sigma^2 = 0, \quad \gamma(i\omega) = 1 \tag{2.4.51}$$

(periodic sampling with skips). In view of

$$\phi_x(\omega) = \frac{2a}{a^2 + \omega^2} \tag{2. 4. 52}$$

and using (2. 4. 47), (2. 4. 37), (2. 4. 38), we obtain

$$\phi_y(\omega) = \frac{1-q}{T} \; \frac{1 - 2qe^{-aT}\cos \omega T + e^{-2aT}(2q-1)}{(1 - e^{-aT}e^{iT\omega})(1 - e^{-aT}e^{-iT\omega})} \tag{2. 4. 53}$$

$$\phi_{xy}(\omega) = \frac{1-q}{T} \; \frac{2a}{a^2 + \omega^2} \; . \tag{2. 4. 54}$$

It may be seen that $\phi_y(\omega)$ can be expressed as

$$\phi_y(\omega) = \frac{1-q}{T} \alpha^2 \; \frac{1 - \mu e^{-iT\omega}}{1 - e^{-aT}e^{-iT\omega}} \cdot \frac{1 - \mu e^{iT\omega}}{1 - e^{-aT}e^{iT\omega}} \tag{2. 4. 55}$$

where $\mu$ is taken to be the smaller root of

$$\mu + \frac{1}{\mu} = \frac{e^{aT} + e^{-aT}(2q-1)}{q} \; {}^{*} \tag{2. 4. 56}$$

and where

$$\alpha^2 = \frac{1 + e^{-2aT}(2q-1)}{1 + \mu^2} \tag{2. 4. 57}$$

With

$$G_1(\omega) = \alpha \sqrt{\frac{1-q}{T}} \; \frac{1 - \mu e^{-i\omega T}}{1 - e^{-aT}e^{-i\omega T}} \; {}^{**} \tag{2. 4. 58}$$

---

* This equation has two real positive roots $\mu_1$ and $\mu_2$ such that $\mu_1 \mu_2 = 1$.

** Observe that $G_1 \neq L_2$ as in the classical theory.

we obtain from (2. 4. 15)

$$H(\omega) = \frac{2a}{\alpha^2} \frac{1 - e^{-aT}e^{-iT\omega}}{1 - \mu e^{-iT\omega}} \int_0^\infty e^{-i\omega T} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-aT}e^{iTu}}{1 - \mu e^{iTu}} \frac{e^{itu}}{u^2 + a^2} du \right\} dt$$

an expression which can be evaluated by residue techniques. The term $\{\ \}$ gives

$$\frac{1}{2a} \frac{1 - e^{-2aT}}{1 - \mu e^{-aT}} e^{-at}$$

and consequently

$$H(\omega) = \frac{1 - e^{-2aT}}{\alpha^2(1 - \mu e^{-aT})} \frac{1 - e^{-aT}e^{-iT\omega}}{a + i\omega} \frac{1}{1 - \mu e^{-i\omega T}} \tag{2. 4. 59}$$

or as an interpolation function

$$h(t) = \frac{1 - e^{-2aT}}{\alpha^2(1 - \mu e^{-aT})} \sum_{n=0}^{\infty} \mu^n k(t + nT) \tag{2. 4. 60}$$

where

$$k(t) = \begin{cases} e^{-at} & , \ 0 < t \leq T \\ 0 & , \ \text{otherwise} \end{cases} \tag{2. 4. 61}$$

the associated minimum mean-square error is calculated from (2. 4. 20) and we find

$$\epsilon^2 = 1 - \frac{(1 - q)(1 - e^{-2aT})^2}{2aT\alpha^2 (1 - \mu e^{-aT})^2} \tag{2. 4. 62}$$

In the absence of skips, the preceding results become ($\mu = 0$)

$$H(\omega) = \frac{1 - e^{-aT}e^{-iT\omega}}{a + i\omega} \tag{2. 4. 63}$$

$$h(t) = \begin{cases} e^{-at} & , \ 0<t \le T \\ 0 & , \ \text{otherwise} \end{cases} \tag{2.4.64}$$

$$\epsilon^2 = 1 - \frac{1 - e^{-2aT}}{2aT} \quad . \tag{2.4.65}$$

In concluding, it is interesting to note that

i) in the absence of skips, the interpolation functions uses only one sample (the most recent one)

ii) in the presence of skips, the interpolation function uses the whole infinite set of past data (the Markovian structure of x(t) has been lost).

2.4.3.2 Poisson sampling with time-jitter and amplitude errors

Referring to (2.2.36), (2.4.34), (2.4.36), we obtain

$$\phi_y(\omega) = \beta(1 + \sigma^2) R_x(0) + \beta^2 \phi_x(\omega) |C(i\omega)|^2 \tag{2.4.66}$$

$$\phi_{xy}(\omega) = \beta \phi_x(\omega) C(i\omega) \tag{2.4.67}$$

Case I: Non-realizable interpolator

In view of (2.4.12), we have

$$H(\omega) = \frac{1}{\beta} \frac{\phi_x(\omega) C(i\omega)}{\phi_x(\omega) |C(i\omega)|^2 + \frac{1 + \sigma^2}{\beta} R_x(0)} \tag{2.4.68}$$

and, as minimum mean-square error, we find

$$\epsilon^2 = \frac{(1 + \sigma^2) R_x(0)}{2\pi} \int_{-\infty}^{\infty} \frac{H(\omega)}{C(i\omega)} \, d\omega \, . \tag{2.4.69}$$

In order to illustrate, suppose that

$$\begin{aligned} R_x(\tau) &= e^{-a|\tau|} \quad , \ a > 0 \\ C(i\omega) &= 1, \quad \sigma^2 = 0 \end{aligned} \tag{2.4.70}$$

(absence of time-jitter and amplitude errors); the preceding equations lead to

$$H(\omega) = \frac{2a}{b^2 + \omega^2} \tag{2.4.71}$$

$$h(t) = \frac{a}{b} e^{-b|t|} \tag{2.4.72}$$

and

$$\epsilon^2 = \frac{a}{b} \tag{2.4.73}$$

where

$$b = + \sqrt{a^2 + 2a\beta} \tag{2.4.74}$$

Case II: Realizable interpolation

We make the same assumptions as in (2.4.70); it follows from (2.4.66), (2.4.67) that

$$\phi_y(\omega) = \beta \frac{b^2 + \omega^2}{a^2 + \omega^2} \tag{2.4.75}$$

$$\phi_{xy}(\omega) = \beta \frac{2a}{a^2 + \omega^2} \tag{2.4.76}$$

where

$$b = + \sqrt{a^2 + 2a\beta} \quad . \tag{2.4.77}$$

With

$$G_1(\omega) = \sqrt{\beta} \quad \frac{\omega - ib}{\omega - ia} \quad ,$$

equation (2.4.15) gives

$$H(\omega) = 2a \frac{\omega - ia}{\omega - ib} \int_0^\infty e^{-i\omega t} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{itu}}{(u + ib)(u - ia)} du \right\} dt$$

and because of

$$\{ \ \} = \frac{e^{-at}}{a + b} \quad ,$$

we obtain

$$H(\omega) = \frac{2a}{a+b} \; \frac{1}{b + i\omega} \tag{2.4.78}$$

$$h(t) = \begin{cases} \dfrac{2a}{a+b}\, e^{-bt} & , \; t > 0 \\ & \\ o & , \; t \quad 0 \end{cases} \tag{2.4.79}$$

with minimum mean-square error

$$\epsilon^2 = 1 - \frac{2a\beta}{(a+b)^2} \quad . \tag{2.4.80}$$

It is interesting to notice that the optimum interpolator is a simple first-order system.

# CHAPTER III

## Step-wise Processes


### Summary

The random telegraph wave and the output of a zero-order hold in a sampled-data control system constitute familiar examples of step-wise processes.

In this chapter, we shall define various models of stationary step-wise processes and we shall investigate their second-order statistics. Time-domain and frequency domain statistics will be considered, with their respective merits.

The results obtained are particularly simple in the case of independent sampling intervals. For illustration, various examples are given.


### 3.1 The chopped random process

#### 3.1.1 Definition

The chopped random process y(t) is a continuous parameter random process defined as

$$y(t) = x(t_n), \quad t_n < t \le t_{n+1} \tag{3.1.1}$$

where x(t) is a continuous parameter stationary random process which is independent of the stationary point process $\{t_n\}$.

Following this definition, y(t) is a stationary random process. This process can be viewed as the output of a zero-order hold, preceded by a sampler which samples (or "chops") the process x(t), at the random instants $t_n$.

For simplicity, we shall assume that the process x(t) is continuous in the mean and has a spectral density denoted by $\phi_x(\omega)$. As for the point process $\{t_n\}$, we shall suppose that equation (1.3.31) is satisfied, that is

$$E[N(\tau)] < \infty^{*} \tag{3.1.2}$$

### 3.1.2 Second-order statistics

First, we observe that

$$E[y(t)] = E[x(t)] \tag{3.1.3}$$

and

$$E[y(t)^2] = R_x(0) \tag{3.1.4}$$

where

$$R_x(t) = E[x(t + \tau)\ x(t)]\ .$$

Denoting by $t_{-1}$ the instant of the most recent occurrence prior to the fixed instant t and using the notations of Chapter I, we can write

$$\begin{aligned} E[x(t+\tau)y(t)] &= E[x(t+\tau)\ x(t_{-1})] \\ &= E[R_x(\tau + L_{-1})] \end{aligned}$$

that is

$$R_{xy}(\tau) = \int_0^\infty R_x(\tau + \sigma)\ g_1(\sigma)d\sigma \tag{3.1.5}$$

where $g_1(\sigma)$ denotes the density function for $L_{-1}$ (or for $L_1$). Using Parseval's relation, equation (3.1.5) can also be written

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_x(\omega)\, g_1^*(-i\omega)\, e^{i\omega\tau}\, d\omega \tag{3.1.6}$$

so that

$$\phi_{xy}(\omega) = g_1^*(-i\omega)\,\phi_n(\omega) \tag{3.1.7}$$

thus giving the cross-spectral density between x(t) and y(t). In an analogous way, we obtain

*This somewhat restrictive assumption is convenient: by virtue of theorem 4, it follows that $\sum_{n=0}^{\infty} p(n, \tau) = 1$, $G_1(0_+) = 0$, $g_1(\sigma) \in L_1(0,\infty)\ L_2(0,\infty), \ldots$ etc. . . .

$$R_{yx}(\tau) = E[y(t+\tau)\, x(t)]$$

$$= \int_0^\infty R_x(\tau - \sigma)\, g_1(\sigma) d\sigma \tag{3.1.8}$$

and

$$\phi_{yx}(\omega) = g_1^*(i\omega)\phi_x(\omega) \tag{3.1.9}$$

Next, let us evaluate the correlation function

$$R_y(\tau) = E[y(t)\, y(t-\tau)] \tag{3.1.10}$$

where without loss of generality, we assume the argument $\tau$ positive. Denoting by $t'_{-1}$ the instant of the most recent occurrence prior to the instant $t - \tau$, we can write

$$R_y(\tau) = E[x(t_{-1})\, x(t'_{-1})]$$

$$= E[R_x(t_{-1} - t'_{-1})]$$

$$= \int_{0-}^\infty R_x(\sigma) dF(\sigma, \tau) \tag{3.1.11}$$

where

$$F(\sigma, \tau) = P[t_{-1} - t'_{-1} \leq \sigma] .$$

This distribution function is determined as follows: letting

$$\psi(\sigma, \tau) = P[(t_{-1} - t'_{-1} \leq \sigma) \cap (L_{-1} \leq \tau)] \tag{3.1.12}$$

$$\psi_n(\sigma, \tau) = P[(L_{-(n+1)} - L_{-1} \leq \sigma) \cap (L_{-n} \leq \tau) \cap (L_{-(n+1)} > \tau) \tag{3.1.13}$$

and observing that

$$P[t_{-1} - t'_{-1} \leq \sigma) \cap (L_{-1} > \tau)] = p(0, \tau) \tag{3.1.14}$$

$$\psi(\sigma, \tau) = \sum_{n=1}^\infty \psi_n(\sigma, \tau) \tag{3.1.15}$$

we then obtain

$$F(\sigma, \tau) = p(0, \tau) + \sum_{n=1}^{\infty} \psi_n(\sigma, \tau) \ , \tag{3.1.16}$$

thus defining $F(\sigma, \tau)$ in terms of the statistics of the point process. It is interesting to notice from (3.1.12), that

$$\psi(\sigma, \tau) \leq \min \begin{cases} G_1(\sigma) \\ G_1(\tau) \end{cases}$$

thus showing that $\psi(\sigma, \tau)$ is continuous at the origin $[\sigma = \tau = 0]$.

The correlation function for the process y(t) is expressed as

$$R_y(\tau) = p(0, \tau) R_x(0) + \int_0^{\infty} R_x(\sigma) \, d\psi(\sigma, \tau) \ , \quad \tau > 0 \tag{3.1.17}$$

where

$$\psi(\sigma, \tau) = \sum_{n=1}^{\infty} P[L_{-(n+1)} - L_{-1} \leq \sigma) \cap (L_{-n} \leq \tau) \cap (L_{-(n+1)} > \tau)] \tag{3.1.18}$$

we may observe that

$$|R_y(\tau) - p(0, \tau) R_x(0)| \leq R_x(0) G_1(\tau)$$

hence

$$R_y(0_+) = R_x(0) = E[y(t)^2] \ ,$$

and so y(t) is continuous in the mean.

3.1.3 Some properties of $\psi(\sigma, \tau)$

i) Denoting

$$\psi'(\sigma, \tau) = \frac{d\psi(\sigma, \tau)}{d\sigma} \qquad \text{a. e.} \tag{3.1.19}$$

$$\psi_n'(\sigma, \tau) = \frac{d\psi_n(\sigma, \tau)}{d\sigma} \qquad \text{a. e.} \tag{3.1.20}$$

it follows from (3.1.15) and from Fubini's derivation theorem (ref R. 2) that

$$\psi'(\sigma, \tau) = \sum_{n=1}^{\infty} \psi_n'(\sigma, \tau) \qquad \text{a. e.} \qquad . \tag{3.1.21}$$

If the $\psi_n'(\sigma, \tau)$ are absolutely continuous in every closed interval, then the same property holds* for $\psi(\sigma, \tau)$.

ii) If the point process $\{t_n\}$ satisfies assumption (2.1.23), namely

$$t_{n+1} - t_n > \ell > 0 \tag{3.1.22}$$

then the summation (3.1.15) is finite and we can write

$$\psi(\sigma, \tau) = \sum_{n=1}^{N(\sigma,\tau)} \psi_n(\sigma, \tau) \tag{3.1.23}$$

where

$$N(\sigma, \tau) = \max \left\{ k \,\middle|\, k \leqslant \min \begin{bmatrix} \frac{\sigma}{\ell} + 1 \\ \frac{\tau}{\ell} + 1 \end{bmatrix} \right\}$$

$$, \quad k = 1, 2, \ldots \tag{3.1.24}$$

iii) Suppose that the point process is such that for some finite positive constant L and for all n

$$t_{n+1} - t_n < L < \infty \tag{3.1.25}$$

with probability one [in other words, the time spacing between consecutive occurrences can not exceed L]. In this case, we may notice that

$$\begin{aligned} \psi(\sigma, \tau) &= 1, \text{ for } \tau > L, \ \sigma > \tau + L \\ \psi(\sigma, \tau) &= G_1(\tau), \text{ for } \tau \leq L, \ \sigma > \tau + L \ . \end{aligned} \tag{3.1.26}$$

iv) An interesting case is provided by assuming that $L_{-1}, x_{-1}, \ldots x_{-n}$ $[x_{-n} = L_{-(n+1)} - L_{-n}]$ are independent random variables such that all the $x_{-n}$ are identically distributed according to density functions and denote

$$f_n(\sigma) = \lim_{h \to 0} P[\sigma < \sum_{k=1}^{n} x_{-k} \leq \sigma + h] \cdot \frac{1}{h} \ . \tag{3.1.27}$$

In this case, the $\psi_n(\sigma, \tau)$ are absolutely continuous and the $\psi_n'(\sigma, \tau)$ can

*This may be seen by combining Beppo-Levi's and Fubini's derivation theorem.

can be evaluated as follows: from the elementary probability

$$P[(\sigma < L_{-2} - L_{-1} \leq \sigma + d\sigma) \cap (x < L_{-1} \leq x + dx) \cap (L_{-2} > \tau)]$$

$$= f_1(\sigma)\,d\sigma\ g_1(x)\,dx\,U(\sigma - \tau + x)^{*}$$

we obtain

$$\psi_1'(\sigma, \tau) = f_1(\sigma) \int_0^{\tau} g_1(x)\ U(\sigma - \tau + x)\,dx\ . \tag{3.1.28}$$

Similarly, from

$$P[(\sigma < L_{-(n+1)} - L_{-1} \leq \sigma + d\sigma) \cap (x < L_{-1} \leq x + dx) \cap$$

$$(v + x < L_{-n} \leq v + dv + x) \cap (L_{-n} \leq \tau) \cap (L_{-(n+1)} > \tau)]$$

$$= P[(x < L_{-1} \leq x + dx) \cap (v < \sum_{k=1}^{n-1} x_{-k} \leq v + dv) \cap$$

$$(\sigma - v < x_{-n} \leq \sigma + d\sigma - v) \cap (L_{-n} \leq \tau) \cap (L_{-(n+1)} > \tau)]$$

$$= g_1(x)\ dx\ f_{n-1}(v)\,dv\,f_1(\sigma - v)\,d\sigma\ U(\tau - x - v)\ U(\sigma - \tau + x)$$

we arrive at

$$\psi_n'(\sigma, \tau) = \int_0^{\tau} g_1(x)\ U(\sigma - \tau + x) \int_0^{\tau - x} f_1(\sigma - v) f_{n-1}(v)\,dv\,dx \tag{3.1.29}$$

$$,\ n \geq 2.$$

By virtue of (3.1.21), we write

$$\psi'(\sigma, \tau) = f_1(\sigma) \int_0^{\tau} g_1(x)\ U(\sigma - \tau + x)\,dx$$

$$+ \sum_{n=2}^{\infty} \int_0^{\tau} g_1(x)\ U(\sigma - \tau + x) \int_0^{\tau - x} f_1(\sigma - v) f_{n-1}(v)\,dv\,dx \quad \text{a. e.} \tag{3.1.30}$$

---

*U denotes the step-function defined by (2.1.33).

and

$$\psi(\sigma, \tau) = \int_0^{\sigma} \psi'(x, \tau)\, dx \tag{3.1.31}$$

### 3.1.4 Examples

#### 3.1.4.1 Periodic sampling

This case is familiar (ref R.1) and the statistics of y(t) are usually approached in the frequency domain (spectral density, etc.). As will be seen, a time domain approach leads to simple results.

Time-domain statistics

The periodic point process is symmetric and has been defined in section 1.4.1. In the present case, the function $\psi(\sigma, \tau)$ is not absolutely continuous and we find

$$F(\sigma, \tau) = \frac{(n+1)T - \tau}{T} U[\sigma - nT] + \frac{\tau - nT}{T} U[\sigma - (n+1)T] \tag{3.1.32}$$

where

$$n = \max\left\{k \middle| k \le \frac{\tau}{T}\right\}, \quad k = 0, 1, 2, \ldots \quad .$$

From equations (3.1.5), (3.1.8) and (3.1.11), we obtain

$$R_{xy}(\tau) = \frac{1}{T}\int_0^T R_x(\tau + \sigma)\, d\sigma \tag{3.1.33}$$

$$R_{yx}(\tau) = \frac{1}{T}\int_0^T R_x(\tau - \sigma)\, d\sigma \tag{3.1.34}$$

$$R_y(\tau) = \frac{(n+1)T - \tau}{T} R_x[nT] + \frac{\tau - nT}{T} R_x[(n+1)T]$$

$$, \quad nT \le \tau \le (n+1)T \tag{3.1.35}$$

This last expression leads to a simple graphical construction of $R_y(\tau)$ [see Fig. 3.1].

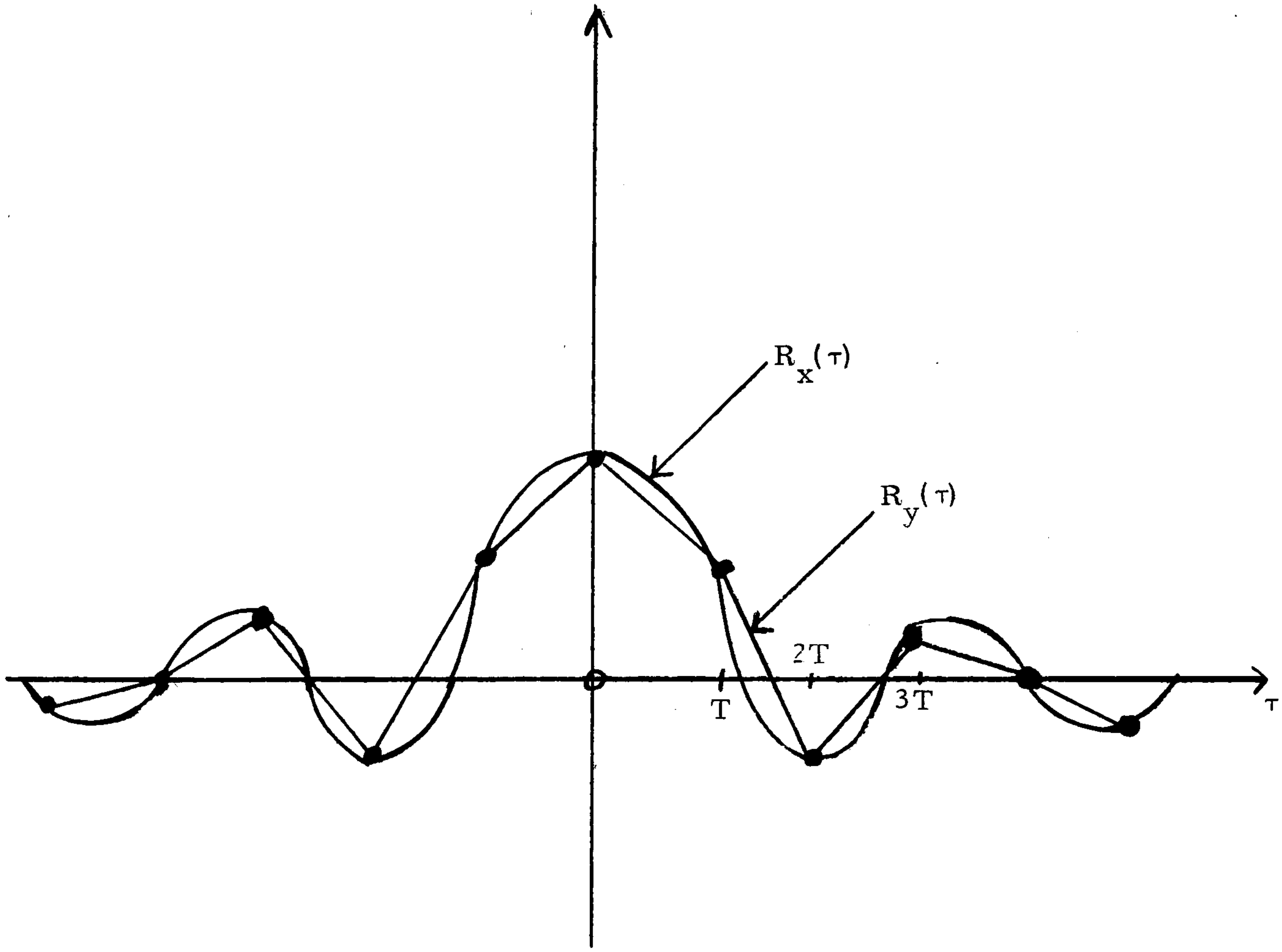


Figure 3.1. Graphical construction of $R_y(\tau)$.

Frequency domain statistics

From equations (3.1.7) and (3.1.9), we obtain

$$\phi_{xy}(\omega) = \frac{1}{T}\,\frac{e^{i\omega T}-1}{i\omega}\,\phi_x(\omega) \tag{3.1.36}$$

$$\phi_{yx}(\omega) = \frac{1}{T}\,\frac{1-e^{-i\omega T}}{i\omega}\,\phi_x(\omega) \quad . \tag{3.1.37}$$

As for the spectral density $\phi_y(\omega)$, it is obtained as follows: letting

$$R_y^*(s) = \int_0^\infty R_y(t)\,e^{-st}dt$$

and using (3.1.35), we have

$$R_y^*(s) = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T}\left\{\frac{(n+1)T-t}{T}R_x[nT] - \frac{t-nT}{T}R_x[(n+1)T]\right\}e^{-st}dt$$

$$= \sum_{n=0}^{\infty} e^{-snT}\int_0^T\left\{\frac{T-t}{T}R_x[nT] + \frac{t}{T}R_x[(n+1)T]\right\}e^{-st}dt$$

$$= \frac{1}{s}R_x(0) + \frac{1}{Ts^2}(e^{-sT}-1)R_x(0) + \frac{(e^{-sT}-1)(1-e^{sT})}{Ts^2}\sum_{n=1}^{\infty}R_x(nT)e^{-snT} \quad .$$

Since

$$\phi_y(\omega) = R_y^*(i\omega) + R^*(-i\omega)$$

we obtain

$$\phi_y(\omega) = \frac{\sin^2\frac{\omega T}{2}}{(\frac{\omega T}{2})^2}\,T\sum_{n=-\infty}^{\infty}R_x(nT)e^{-in\omega T} \tag{3.1.38}$$

or in view of equation (2.4.47)

$$\phi_y(\omega) = \frac{\sin^2\frac{\omega T}{2}}{(\frac{\omega T}{2})^2}\sum_{n=-\infty}^{\infty}\phi_x(\omega - n\omega_0) \tag{3.1.39}$$

a result which is expected (ref R. 1).

Illustration: If we take

$$R_x(\tau) = e^{-a|\tau|} \quad , \; a > 0$$

that is

$$\phi_x(\omega) = \frac{2a}{a^2 + \omega^2} \, ,$$

the preceding results lead to

$$R_{xy}(\tau) = \begin{cases} \dfrac{1 - e^{-aT}}{aT} e^{-a\tau} & , \; \tau \geq 0 \\[2ex] \dfrac{2 - e^{a\tau} - e^{-a\tau} e^{-aT}}{aT} & , \; -T \leq \tau \leq 0 \\[2ex] \dfrac{e^{aT} - 1}{aT} e^{a\tau} & , \; \tau \leq -T \end{cases}$$

$$R_y(\tau) = \frac{(n+1)T - \tau}{T} e^{-anT} + \frac{\tau - nT}{T} e^{-a(n+1)T} \quad , \; nT \leq \tau \leq (n+1)T$$

and

$$\phi_{xy}(\omega) = \frac{1}{T} \cdot \frac{e^{i\omega T} - 1}{i\omega} \cdot \frac{2a}{a^2 + \omega^2}$$

$$\phi_y(\omega) = \frac{\sin^2 \frac{\omega T}{2}}{(\frac{\omega T}{2})^2} \cdot \frac{T(1 - e^{-2aT})}{1 - 2e^{-aT} \cos \omega T + e^{-2aT}} \; .$$

### 3.1.4.2 Poisson sampling time-domain statistics

From equation (3.1.5), (3.1.8) we obtain the cross-correlation functions

$$R_{xy}(\tau) = \beta \int_0^\infty R_x(\tau + \sigma) e^{-\beta\sigma} d\sigma \tag{3.1.40}$$

$$R_{yx}(\tau) = \beta \int_0^\infty R_x(\tau - \sigma) e^{-\beta\sigma} d\sigma \; . \tag{3.1.41}$$

In order to determine $R_y(\tau)$, we need first to evaluate $\psi(\sigma, \tau)$ or $\psi'(\sigma, \tau)$*

*Here, $\psi$ is absolutely continuous.

and we shall indicate two ways of calculation:

i) Letting

$$L'_{-1} = t - \tau - t'_{-1}$$

and observing that

$$P[(x < L_{-1} \le x + dx) \cap (\sigma < t_{-1} - t'_{-1} \le \sigma + d\sigma)]$$

$$= P[(x < L_{-1} \le x + dx) \cap (\sigma + x - \tau < L'_{-1} \le \sigma + d\sigma + x - \tau)]$$

$$= P[x < L_{-1} \le x + dx]\, P[\sigma + x - \tau < L'_{-1} \le \sigma + d\sigma + x - \tau]$$

$$= g_1(x)\,dx\; g_1(\sigma - \tau + x)\,d\sigma \quad *$$

we obtain from (3.1.12)

$$\psi'(\sigma, \tau) = \int_0^\tau g_1(x)\, g_1(\sigma - \tau + x)\,dx$$

$$\psi'(\sigma,\tau) = \beta^2 e^{-\beta(\sigma-\tau)} \int_0^\tau e^{-2\beta x} U(\sigma - \tau + x)\,dx$$

that is

$$\psi'(\sigma,\tau) = \begin{cases} \frac{1}{2}\beta e^{-\beta\tau}(e^{\beta\sigma} - e^{-\beta\sigma}) & , \; 0 \le \sigma \le \\ \frac{1}{2}\beta e^{-\beta\sigma}(e^{\beta\tau} - e^{-\beta\tau}) & , \; \sigma > \tau \end{cases} \tag{3.1.42}$$

ii) The same result is obtained by evaluation of the infinite series defined by equation (3.1.30). The function $\psi'(\sigma,\tau)$ is sketched in Fig. 3.2.

From equation (3.1.17), we now obtain the auto-correlation function for y(t) as

$$R_y(\tau) = e^{-\beta\tau} R_x(0) + \tfrac{1}{2}\beta e^{-\beta\tau} \int_0^\tau R_x(\sigma)(e^{\beta\sigma} - e^{-\beta\sigma})\,d\sigma$$

$$+ \tfrac{1}{2}\beta(e^{\beta\tau} - e^{-\beta\tau}) \int_\tau^\infty R_x(\sigma) e^{-\beta\sigma}\,d\sigma \;, \quad \tau \ge 0 \tag{3.1.43}$$

*The Poisson point process has no memory; $L_{-1}$ and $L'_{-1}$ are independent and identically distributed.

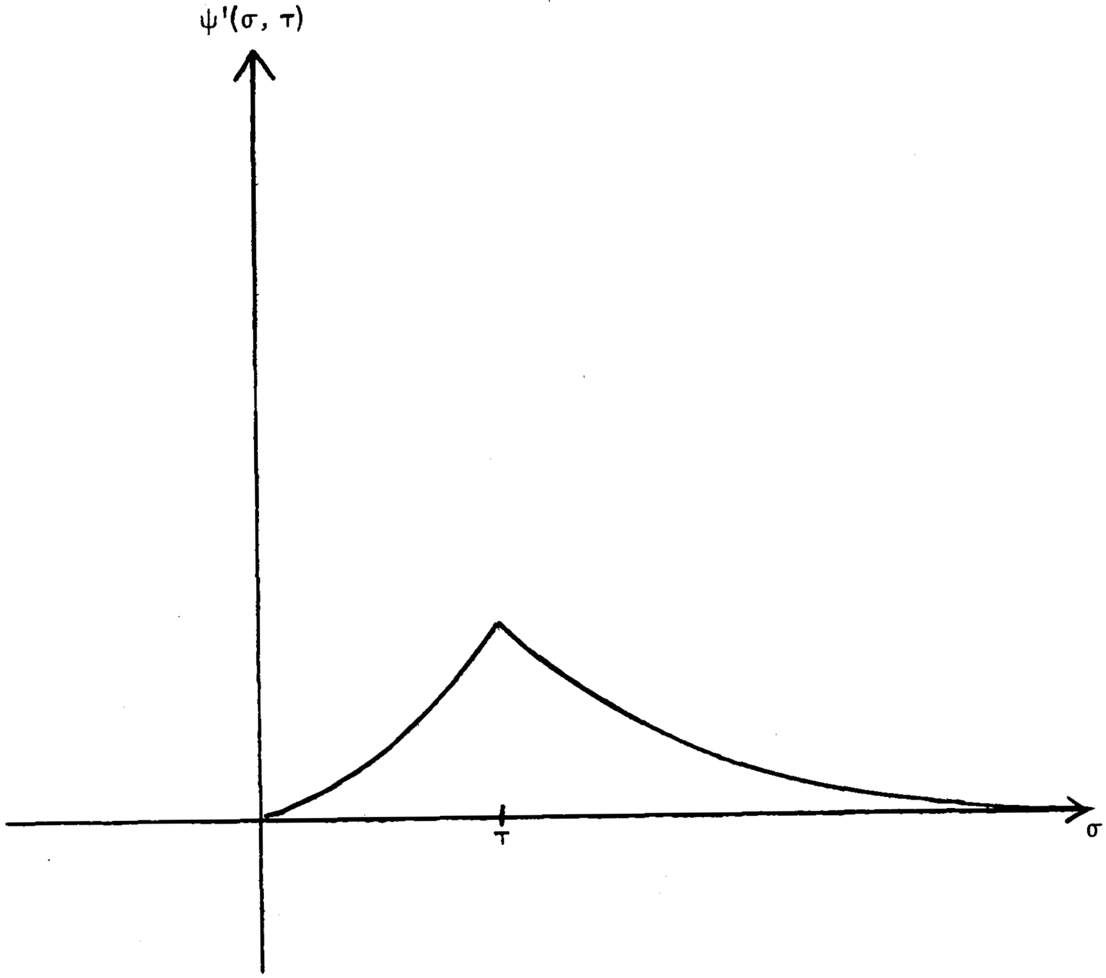


Figure 3. 2. The function $\psi'(\sigma, \tau)$, for a Poisson point process.

Frequency-domain statistics

Using equations (3.1.7) and (3.1.9), we have

$$\phi_{xy}(\omega) = \frac{\beta}{\beta - i\omega}\,\phi_x(\omega) \qquad (3.1.44)$$

$$\phi_{yx}(\omega) = \frac{\beta}{\beta + i\omega}\,\phi_x(\omega) \quad . \qquad (3.1.45)$$

The spectral density $\phi_y(\omega)$ can be evaluated from (3.1.43); letting

$$R_y^*(s) = \int_0^\infty e^{-s\tau} R_y(\tau)\,d\tau , \quad Re(s) > 0$$

we find that

$$R_y^*(s) = \frac{R_x(0)}{s + \beta} + \frac{\beta^2}{\beta^2 - s^2} R_x^*(s) - \frac{\beta^2}{\beta^2 - s^2} R_x^*(\beta) . \qquad (3.1.46)$$

consequently

$$\phi_y(\omega) = \frac{\beta^2}{\beta^2 + \omega^2}\left[\phi_x(\omega) + \phi_n(\omega)\right] \qquad (3.1.47)$$

where

$$\phi_n(\omega) = 2\int_0^\infty e^{-\beta\tau}[R_x(0) - R_x(\tau)]\,d\tau = \text{const.} \geq 0 \qquad (3.1.48)$$

In other words, the spectral density is the same as the one obtained at the output of a first-order system, with transfer function $\frac{\beta}{\beta + s}$, with input $x(t) + n(t)$, where $n(t)$ is a white-noise with spectral density $\phi_n(\omega)$ and zero-mean, and $x(t)$ and $n(t)$ are uncorrelated.

Illustration: Taking

$$R_x(\tau) = e^{-a|\tau|} , \quad a > 0$$

we obtain for $\beta \neq a$

$$R_{xy}(\tau) = \begin{cases} \dfrac{\beta}{a + \beta}\, e^{-a\tau} & , \ \tau \geq 0 \\[2ex] \dfrac{\beta}{a + \beta}\, e^{\beta\tau} + \dfrac{\beta}{a - \beta}\left[e^{\beta\tau} - e^{a\tau}\right] & , \ \tau \leq 0 \end{cases}$$

$$R_y(\tau) = \frac{1}{\beta^2 - a^2} (\beta^2 e^{-a|\tau|} - a^2 e^{-\beta|\tau|})$$

and for $\beta = a$

$$R_{xy}(\tau) = \begin{cases} \frac{1}{2} e^{-a\tau} & , \quad \tau \geq 0 \\ \frac{1}{2} e^{a\tau} - a\tau e^{a\tau} & , \quad \tau \leq 0 \end{cases}$$

$$R_y(\tau) = e^{-a|\tau|} + \frac{1}{2} a|\tau| e^{-a|\tau|} \quad .$$

In the frequency domain, we obtain

$$\phi_{xy}(\omega) = \frac{\beta}{\beta - i\omega} \frac{2a}{a^2 + \omega^2}$$

and

$$\phi_y(\omega) = \frac{\beta^2}{\beta^2 + \omega^2} \left[ \frac{2a}{a^2 + \omega^2} + \frac{2a}{\beta(a + \beta)} \right] .$$

### 3.1.5 Frequency analysis

#### 3.1.5.1 The spectral density

For simplicity, let us assume* that the $\psi_n(\sigma, \tau)$ are absolutely continuous on $[0, \infty)$; it follows from (3.1.17) and (3.1.19) that

$$R_y(\tau) = p(0, \tau)\, R_x(0) + \int_0^\infty R_x(\sigma)\, \psi'(\sigma, \tau) d\sigma \quad , \quad \tau \geq 0 \quad . \qquad (3.1.49)$$

---

* Otherwise, a Fourier-Stieltjes theory would be necessary; however, if the $\psi_n(\sigma; \tau)$ have no singular component (which arises from the Lebesgue decomposition of a monotone function) and if the $\psi'_n(\sigma, \tau)$ and $\psi'(\sigma, \tau)$ denote generalized derivatives, then equation (3.1.49) and the following ones remain valid by using the concept of a generalized function and of a generalized Fourier theory (ref L. 4, E. 1, E. 2, L. 2, S. 1, ...).

Using the Parseval relation, equation (3.1.49) can be written

$$R_y(\tau) = p(0,\tau)\,R_x(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_x(u)\psi'^*(-iu,\tau)\,du\ ,\quad \tau \geq 0 \qquad (3.1.50)$$

where

$$\psi'^*(-iu,\tau) = \int_0^{\infty} e^{iu\sigma}\,\psi(\sigma,\tau)\,d\sigma\ .$$

In view of

$$\int_0^{\infty} e^{-Re(s)\tau}\int_{-\infty}^{\infty}\phi_x(u)\,du\,d\tau = 2\pi\int_0^{\infty} R_x(0)\,e^{-Re(s)\tau}\,d\tau < \infty\quad ,\quad Re(s) > 0$$

we observe that the iterated integral

$$\int_0^{\infty} e^{-s\tau}\int_{-\infty}^{\infty}\phi_x(u)\psi'^*(-iu,\tau)\,du\,d\tau\quad ,\quad Re(s) > 0$$

is absolutely convergent; therefore, the order of integrations may be interchanged*. As a result, from equation (3.1.50), we obtain

$$R_y^*(s) = p^*(0,s)\,R_x(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_x(u)\psi'^{**}(-iu,s)\,du\quad ,\ Re(s) > 0 \qquad (3.1.51)$$

where

$$R_y^*(s) = \int_0^{\infty} e^{-s\tau} R_y(\tau)\,d\tau \qquad (3.1.52)$$

and

$$\psi'^{**}(-iu,s) = \int_0^{\infty} e^{-s\tau}\int_0^{\infty} e^{iu\sigma}\,\psi(\sigma,\tau)\,d\sigma\ \ d\tau \qquad (3.1.53)$$

Finally, in view of

$$\phi_y(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\,R_y(\tau)\,d\tau$$

the spectral density $\phi_y(\omega)$ is obtained from

$$\phi_y(\omega) = R_y^*(i\omega) + R_y^*(-i\omega)\quad . \qquad (3.1.54)$$

---

*Tonelli-Hobson's theorem (ref G. 2, page 3).

General remarks:

The preceding expressions for $\phi_y(\omega)$ are formal since $R_y^*(i\omega)$ and hence $\phi_y(\omega)$ might not exist in the usual function sense, but will have to be interpreted in a generalized function sense.

In view of a possible evaluation of (3.1.51) by residue methods, it is interesting to notice that, in the u-plane, the upper-half plane does not contain any poles of $\psi'^{**}(-iu, s)$ but only poles of $\phi_x(u)$; in the lower half-plane, we find all the poles of $\psi'^{**}(iu, s)$, in addition to poles arising from $\phi_x(u)$.

Because of

$$\int_0^\infty \psi'_n(\sigma, \tau)\, d\sigma = p(n, \tau) \leq 1 ,$$

the iterated integral

$$\psi'^{**}(-iu, s) = \int_0^\infty e^{-s\tau} \int_0^\infty e^{iu\sigma} \psi'_n(\sigma, \tau)\, d\sigma\, d\tau \quad , \quad \mathrm{Re}(s) > 0$$

is absolutely convergent; it then follows* that

$$\psi'^{**}_n(-iu, s) = \int_0^\infty e^{iu\sigma} \int_0^\infty e^{-s\tau} \psi'_n(\sigma, \tau)\, d\tau\, d\sigma = \int_0^\infty \int_0^\infty e^{-s\tau} e^{iu\sigma} \psi'_n(\sigma, \tau)\, d(\sigma, \tau) ,$$

where the double integral is absolutely convergent. From Lebesgue's convergence theorem, it is easily seen that

$$\psi'^{**}(-iu, s) = \sum_{n=1}^{\infty} \psi'^{**}_n(-iu, s) , \quad \mathrm{Re}(s) > 0 . \qquad (3.1.55)$$

3.1.5.2 The case of independent sampling intervals

We shall evaluate $\psi'^{**}(-iu, s)$ and $R_y^*(s)$ under the assumption that $L_{-1}, x_{-1}, \ldots x_{-n}$ are independent random variables and that all the $x_{-n}$ are identically distributed with density function $f_1(\sigma)$.

*Tonnelli-Hobson's theorem (ref G. 2).

In view of (3.1.28), we have

$$\psi_1'^{*}(-iu, \tau) = \int_0^\infty e^{iu\sigma} f_1(\sigma) \int_0^\tau g_1(x) U(\sigma - \tau + x)\, dx\, d\sigma$$

and observing that this iterated integral is absolutely convergent, we obtain, by interchanging the order of integrations, that

$$\psi_1'^{*}(-iu, \tau) = \int_0^\tau g_1(x) \int_0^\infty e^{iu\sigma} f_1(\sigma)\, d\sigma\, dx - \int_0^\tau g_1(x) \int_0^{\tau - x} e^{iu\sigma} f_1(\sigma)\, d\sigma\, dx$$

$$= f_1^*(-iu) \int_0^\tau g_1(x)\, dx - \int_0^\tau g_1(x) \int_0^{\tau-x} e^{iu\sigma} f_1(\sigma)\, d\sigma\, dx \quad ;$$

then

$$\psi_1'^{**}(-iu, s) = f_1^*(-iu) \int_0^\infty e^{-s\tau} \int_0^\tau g_1(x)\, dx\, d\tau - \int_0^\infty e^{-s\tau} \int_0^\tau g_1(x) \int_0^{\tau-x} e^{iu\sigma} f_1(\sigma)\, d\sigma\, dx\, d\tau$$

$$= f_1^*(-iu) \frac{g_1^*(s)}{s} - g_1^*(s) \frac{f_1^*(s - iu)}{s} \quad .$$

In view of the fact that for independent sampling intervals

$$g_1^*(s) = \beta \frac{1 - f_1^*(s)}{s} \,,$$

where

$$\beta = \frac{1}{\int_0^\infty x f_1(x)\, dx} \qquad (3.1.57)$$

we obtain

$$\psi_1'^{**}(-iu, s) = \beta \frac{1 - f_1^*(s)}{s^2} [f_1^*(-iu) - f_1^*(s - iu)], \quad \mathrm{Re}(s) > 0 \qquad (3.1.58)$$

For $n > 1$, from (3.1.29), we have

$$\psi_n'^*(-iu, \tau) = \int_0^\infty e^{iu\sigma}\int_0^\tau g_1(x)U(\sigma - \tau + x)\int_0^{\tau - x} f_1(\sigma - v)f_{n-1}(v)\,dv\,dx\,d\sigma \quad ;$$

noticing that the iterated integrals

$$\int_0^\tau g_1(x)\int_{\tau-x}^\infty e^{iu\sigma}\int_0^{\tau-x} f_1(\sigma - v)f_{n-1}(v)\,dv\,d\sigma\,dx$$

$$\int_0^\tau g_1(x)\int_0^{\tau-x} f_{n-1}(v)\int_{\tau-x}^\infty f_1(\sigma - v)e^{iu\sigma}\,d\sigma\,dv\,dx$$

are absolutely convergent, we may therefore interchange the order of integrations and we obtain

$$\psi_n'^*(-iu, \tau) = \int_0^\tau g_1(x)\int_0^{\tau-x} f_{n-1}(v)\int_0^\infty f_1(\sigma - v)e^{iu\sigma}\,d\sigma\,dv\,dx$$

$$- \int_0^\tau g_1(x)\int_0^{\tau-x} f_{n-1}(v)\int_0^{\tau-x} f_1(\sigma - v)e^{iu\sigma}\,d\sigma\,dv\,dx\,.$$

Observing that for $v \geq 0$

$$\int_0^\infty f_1(\sigma - v)e^{iu\sigma}\,d\sigma = \int_0^\infty f_1(t)e^{iu(t+v)}\,dt = e^{iuv}f_1^*(-iu)$$

and that

$$\int_0^{\tau-x} f_{n-1}(v)\int_0^{\tau-x} f_1(\sigma - v)e^{iu\sigma}\,d\sigma\,dv = \int_0^{\tau-x} e^{iu\sigma}\int_0^{\tau-x} f_1(\sigma - v)f_{n-1}(v)\,dv\,d\sigma$$

$$= \int_0^{\tau-x} e^{iu\sigma}\int_0^\infty f_1(\sigma - v)f_{n-1}(v)\,dv\,d\sigma$$

we can write

$$\psi_n'^{*}(-iu, \tau) = f_1^*(-iu)\int_0^{\tau} g_1(x)\int_0^{\tau-x} e^{iuv} f_{n-1}(v)\,dv\,dx$$

$$- \int_0^{\tau} g_1(x)\int_0^{\tau-x} e^{iu\sigma}\int_0^{\infty} f_1(\sigma - v) f_{n-1}(v)\,dv\,d\sigma\,dx\ .$$

Then

$$\psi_n'^{**}(-iu, s) = f_1^*(-iu)\int_0^{\infty} e^{-s\tau}\int_0^{\tau} g_1(x)\int_0^{\tau-x} e^{iuv} f_{n-1}(v)\,dv\,dx\,d\tau$$

$$- \int_0^{\infty} e^{-s\tau}\int_0^{\tau} g_1(x)\int_0^{\tau-x} e^{iu\sigma}\int_0^{\infty} f_1(\sigma - v) f_{n-1}(v)\,dv\,d\sigma\,dx\,d\tau$$

and using convolution properties, we find

$$\psi_n'^{**}(-iu, s) = f_1^*(-iu) g_1^*(s)\ \frac{f_{n-1}^*(s - iu)}{s} - g_1^*(s)\ \frac{f_1^*(s - iu) f_{n-1}^*(s - iu)}{s}\ ,$$

that is

$$\psi_n'^{**}(-iu, s) = \beta\ \frac{1 - f_1^*(s)}{s^2}[f_1^*(-iu) - f_1^*(s - iu)]\left\{f_1^*(s - iu)\right\}^{n-1}$$

$$, \mathrm{Re}(s) > 0\ . \qquad (3.1.59)$$

Summation of the infinite series (3.1.55) gives

$$\psi'^{**}(-iu, s) = \beta\,\frac{1 - f_1^*(s)}{s^2}\ \frac{f_1^*(-iu) - f_1^*(s - iu)}{1 - f_1^*(s - iu)}\ , \quad \mathrm{Re}(s) > 0 \qquad (3.1.60)$$

Finally, in view of

$$R_x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_x(u)\,du$$

and

$$p^*(0, s) = \frac{1 - g_1^*(s)}{s}$$

we obtain from equation (3.1.51) that

$$R_y^*(s) = \frac{R_x(0)}{s} + \beta\,\frac{1 - f_1^*(s)}{2\pi s^2}\int_{-\infty}^{\infty}\phi_x(u)\,\frac{f_1^*(-iu) - 1}{1 - f_1^*(s - iu)}\,du\ , \quad Re(s) > 0\ . \tag{3.1.61}$$

Remarks:

In the preceding calculations, it was assumed that $f_1(\sigma)$ is an ordinary density function. In a generalized Laplace transform theory, these calculations can be extended to include cases where $f_1(\sigma)$ is a generalized density function which contains $\delta$-functions. Also, we may observe that $\frac{1}{1 - f_1^*(s - iu)}$ has only one pole $u = -is$ if $x_{-n}$ has a non-zero continuous component in its distribution function and that it has an infinite number of poles if $x_{-n}$ has a lattice distribution (ref L. 5, L. 6).

Examples:

i) If we take $R_x(\tau) = 1$, or as generalized spectral density $\phi_x(u) = 2\pi\delta(u)$, we obtain

$$R_y^*(s) = \frac{1}{s}\ ,$$

as it should be.

ii) If $x(t)$ is such that

$$R_x(\tau) = e^{-a|\tau|}\ , \quad a > 0 \tag{3.1.62}$$

or

$$\phi_x(u) = \frac{2a}{a^2 + u^2}\ ,$$

evaluation of (3.1.61) by residue methods leads to

$$R_y^*(s) = \frac{1}{s} + \beta\,\frac{1 - f_1^*(s)}{s^2}\,\frac{f_1^*(a) - 1}{1 - f_1^*(s + a)}\ , \quad Re(s) > 0\ . \tag{3.1.63}$$

For illustration, if we let

$$f_1^*(s) = e^{-sT}\ , \quad \beta = \frac{1}{T} \quad \text{(periodic sampling)}$$

or

$$f_1^*(s) = \frac{\beta}{\beta + s}\ , \quad \text{(Poisson sampling)}$$

the results obtained agree with sections 3.1.4.1 and 3.1.4.2.

iii) Finally, if

$$R_x(\tau) = \cos \omega_0 \tau \tag{3.1.64}$$

or

$$\phi_x(u) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

then

$$R_y^*(s) = \frac{1}{s} + \beta \frac{1 - f_1^*(s)}{2s^2}\left[\frac{f_1^*(-i\omega_0) - 1}{1 - f_1^*(s - i\omega_0)} + \frac{f_1^*(i\omega_0) - 1}{1 - f_1^*(s + i\omega_0)}\right] \quad , \ \mathrm{Re}(s) > 0 \ . \tag{3.1.65}$$

3.1.5.3 <u>An alternative expression for the spectrum</u>

Here, we shall suppose that we can find two positive numbers r and v such that

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-v\tau} \int_0^{\infty} e^{r\sigma} \psi_n'(\sigma, \tau)\, d\sigma\, d\tau < \infty \tag{3.1.66}$$

or, what is equivalent, that

$$\int_0^{\infty} e^{-v\tau} \int_0^{\infty} e^{r\sigma} \psi'(\sigma, \tau)\, d\sigma\, d\tau < \infty \ . \tag{3.1.67}$$

Before going on, we may first note that (3.1.66) or (3.1.67) are always true with $r = 0$, $v > 0$. Let us mention that assumption (3.1.66) (or 3.1.67) [with $r > 0$, $v \gtrsim 0$] is not unduly restrictive: in fact,

i) If the point process $\{t_n\}$ is such that equation (3.1.25) holds with probability one [the sampling intervals are bounded from above], then equation (3.1.67) can be written

$$\int_0^{\infty} e^{-v\tau} \int_0^{\tau+L} e^{r\sigma} \psi'(\sigma, \tau)\, d\sigma\, d\tau \le \int_0^{\infty} e^{-v\tau} e^{r(\tau+L)}\, d\tau \ .$$

It then follows that condition (3.1.67) will be fulfilled if we take any positive numbers r and v such that

$$0 < r < v < \infty . \tag{3.1.68}$$

ii) If the sampling intervals are independent random variables and if there exists a positive number $\lambda$ such that*

$$\int_0^\infty e^{\lambda\sigma} f_1(\sigma)\, d\sigma < \infty \tag{3.1.69}$$

then it can be seen by calculations similar to the ones of section 3.1.5.2 that condition (3.1.66) is fulfilled by taking any positive numbers r and v such that

$$0 < r \le \lambda , \quad r < v \quad . \tag{3.1.70}$$

From equation (3.1.49), we obtain

$$R_y^*(s) = p^*(0, s) R_x(0) + \int_0^\infty e^{-s\tau} \int_0^\infty R_x(\sigma)\psi'(\sigma, \tau)\, d\sigma\, d\tau \quad , \qquad Re(s) > 0$$

and, by virtue of

$$\int_0^\infty e^{-Re(s)\tau} \int_0^\infty |R_x(\sigma)| \psi'(\sigma, \tau)\, d\sigma\, d\tau \le \int_0^\infty e^{-Re(s)\tau} R_x(0)\, d\tau < \infty ,$$

we may interchange the order of integrations, so that

$$R_y^*(s) = p^*(0, s) R_x(0) + \int_0^\infty R_x(\sigma)\psi'^*(\sigma, s)\, d\sigma \quad , \qquad Re(s) > 0 \tag{3.1.71}$$

where

$$\psi'^*(\sigma, s) = \int_0^\infty e^{-s\tau} \psi'(\sigma, \tau)\, d\tau .$$

Letting

$$R_x^*(p) = \int_0^\infty R_x(\sigma) e^{-p\sigma}\, d\sigma \tag{3.1.72}$$

$$p = r + iu, \quad r > r_R$$

*For the Poisson point process, $\lambda < \beta$.

where

$$r_R \leq 0 \qquad (3.1.73)$$

denotes the abscissa of absolute convergence for (3.1.72) and

$$\psi'^{**}(-p, s) = \int_0^\infty e^{-s\tau} \int_0^\infty e^{p\sigma} \psi'(\sigma, \tau)\, d\sigma\, d\tau$$

$$= \int_0^\infty e^{p\sigma} \int_0^\infty e^{-s\tau} \psi'(\sigma, \tau)\, d\tau\, d\sigma$$

$$p = r + iu \quad , \quad s = v + i\omega \ , \qquad (3.1.74)$$

We write, in view of the Parseval relation that

$$\int_0^\infty [R_x(\sigma) e^{-r\sigma}][\psi'^{*}(\sigma, s) e^{r\sigma}] d\sigma = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x^*(r + iu) \psi'^{**}(-r - iu, s)\, du$$

and therefore

$$R_y^*(s) = p^*(0, s) R_x(0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x^*(p) \psi'^{**}(-p, s)\, du$$

$$p = r + iu \quad , \quad s = v + i\omega \quad . \qquad (3.1.75)$$

Remark:

The preceding expression resembles equation (3.1.51). However, it is interesting to notice a certain advantage of equation (3.1.75): in the p-plane, the left half plane $\mathrm{Re}(p) < r$ does not contain any poles of $\psi'^{**}(-p, s)$; in the right half plane $\mathrm{Re}(p) > r$, we do not find any poles of $R^*(p)$. This remark will be useful whenever the integral of (3.1.75) can be evaluated by residue methods.

The case of independent sampling intervals

A calculation similar to the one performed in section (3.1.5.2) leads to

$$R_y^*(s) = \frac{1 - g_1^*(s)}{s} R_x(0) + \frac{g_1^*(s)}{2\pi s} \int_{-\infty}^{\infty} R_x^*(p) \frac{f_1^*(-p) - f_1^*(s - p)}{1 - f_1^*(s - p)}\, du \qquad (3.1.76)$$

where

$$g_1^*(s) = \beta \frac{1 - f_1^*(s)}{s}$$

$$p = r + iu , \quad s = v + i\omega$$

$$0 < r < \lambda , \quad r < v$$

Note: $\lambda$ has been defined in (3.1.69).

Examples:

i) Poisson sampling

With equation (3.1.76) and

$$f_1^*(s) = \frac{\beta}{\beta + s} \quad ,$$

we arrive at

$$R_y^*(s) = \frac{1}{s + \beta} R_x(0) + \frac{\beta^2}{\beta + s} \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x^*(p) \frac{1}{(p - s)(p - \beta)} du \quad .$$

Evaluation by residues gives

$$R_y^*(s) = \frac{R_x(0)}{s + \beta} + \frac{\beta^2}{\beta^2 - s^2} R_x^*(s) - \frac{\beta^2}{\beta^2 - s^2} R_x^*(\beta)$$

which is the same as (3.1.46).

ii) Periodic sampling with skips

Here, we have

$$f_1^*(s) = \frac{(1 - q)e^{-sT}}{1 - q e^{-sT}} \tag{3.1.77}$$

and

$$g_1^*(s) = \frac{1 - q}{T} \frac{1 - e^{-sT}}{s(1 - qe^{-sT})} \quad . \tag{3.1.78}$$

From (3.1.76), we are led to

$$R_y^*(s) = \frac{1 - g_1^*(s)}{s} R_x(0) + (1-q) g_1^*(s) \frac{1 - e^{-sT}}{2\pi s} \int_{-\infty}^{\infty} \frac{R_x^*(p)}{(e^{-pT} - q)(1 - e^{(p-s)T})} du$$

$$p = r + iu \quad , \quad s = v + i\omega$$

$$o < r < \lambda \quad , \quad r < v$$

where

$$e^{-\lambda T} = q \quad . \tag{3.1.79}$$

The preceding expression can be evaluated by residue methods; first, we may notice the strong convergence to zero of the integrand as $|p| \to \infty$ along any ray which makes an angle $< \frac{\pi}{2}$ with the real axis; then, we observe that in the right half-plane $\mathrm{Re}(p) > r$, we have a double infinity of poles defined as

$$p = \lambda + n\, i\omega_0$$

$$p = s + n\, i\omega_0$$

where $\omega_0 = \frac{2\pi}{T}$ and $n = 0, \pm 1, \pm 2, \ldots$ . Performing this residue evaluation and using equation (3.1.78), we finally arrive at

$$R_y^*(s) = \frac{1 - g_1^*(s)}{s} R_x(0) + g_1^*(s) g_1^*(-s) \left\{ \sum_{n=-\infty}^{\infty} R_x^*(s + in\omega_0) - \sum_{n=-\infty}^{\infty} R_x^*(\lambda + in\omega_0) \right\} \tag{3.1.80}$$

or in terms of spectral density

$$\phi_y(\omega) = \frac{\sin^2 \frac{\omega T}{2}}{(\frac{\omega T}{2})^2} \frac{1}{1 - 2q \cos \omega T + q^2} \left\{ (1-q)^2 \sum_{n=-\infty}^{\infty} \phi_x(\omega - n\omega_0) \right.$$

$$\left. + (1 - q^2) T R_x(0) - 2(1-q)^2 \sum_{n=-\infty}^{\infty} R_x^*(\lambda + in\omega_0) \right\} . \tag{3.1.81}$$

In view of the Poisson summation formula (ref D. 3, Z. 1),

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} R_x^*(\lambda + in\omega_0) = \frac{R_x(0)}{2} + \sum_{n=1}^{\infty} R_x(nT) e^{-\lambda n T}$$

and because of (3.1.79), equation (3.1.81) can be written as

$$\phi_y(\omega) = \frac{\sin^2 \frac{\omega T}{2}}{(\frac{\omega T}{2})^2} \frac{1}{1 - 2q \cos \omega T + q^2} \{(1 - q)^2 \sum_{n=-\infty}^{\infty} \phi_x(\omega - n\omega_0)$$

$$+ 2q(1 - q) T R_x(0) - 2(1 - q)^2 T \sum_{n=1}^{\infty} R_x(nT) q^n\} \tag{3.1.82}$$

Illustration: if x(t) is such that

$$R_x(\tau) = e^{-a|\tau|}, \qquad a > 0$$

we obtain from (3.1.82) that

$$\phi_y(\omega) = T \frac{\sin^2 \frac{\omega T}{2}}{(\frac{\omega T}{2})^2} \frac{1}{1 - 2q \cos \omega T + q^2} \{ \frac{(1 - q)^2 (1 - e^{-2aT})}{1 - 2e^{-aT} \cos \omega T + e^{-2aT}}$$

$$+ 2q(1 - q) \frac{1 - e^{-aT}}{1 - qe^{-aT}} \} .$$

Remark:

In the absence of skips ($q = 0$), equation (3.1.82) gives

$$\phi_y(\omega) = \frac{\sin^2 \frac{\omega T}{2}}{(\frac{\omega T}{2})^2} \sum_{n=-\infty}^{\infty} \phi_x(\omega - n\omega_0)$$

as it should be.

## 3.2 The chopped and alternated random process

### 3.2.1 Definition

The chopped and alternated random process y(t) is a continuous parameter process defined as

$$y(t) = (-1)^{n+j} x(t_n), \quad t_n < t \leq t_{n+1} \tag{3.2.1}$$

where j is a random variable taking on the values zero or one with equal probability (independently of x(t) and $\{t_n\}$) and where x(t), $\{t_n\}$ are defined as in section 3.1.1.

### 3.2.2 Second-order statistics

First, we may note that

$$E[y(t)] = 0 \tag{3.2.2}$$

$$R_{xy}(\tau) = R_{yx}(\tau) = 0 \tag{3.2.3}$$

and

$$E[y(t)^2] = R_x(0) \quad . \tag{3.2.4}$$

Using the notations of section (3.1.2) and definition (1.2.11), and observing that

$$E[y(t)y(t-\tau)/A_n(t-\tau,\tau)] = (-1)^n E[x(t_{-1})x(t'_{-1})/A_n(t-\tau,\tau)]$$

we obtain as autocorrelation function

$$R_y(\tau) = p(0,\tau)R_x(0) + \sum_{n=1}^{\infty}(-1)^n \int_0^{\infty} R_x(\sigma)\,d\psi_n(\sigma,\tau), \qquad \tau \geq 0 \tag{3.3.5}$$

where the $\psi_n(\sigma, \tau)$ are defined by equation (3,1.13).

Example: Periodic sampling

As in section 3.1.4.1, we arrive at

$$R_y(\tau) = (-1)^n \left\{ \frac{(n+1)T - \tau}{T} R_x[nT] - \frac{\tau - nT}{T} R_x[(n+1)T] \right\}$$

$$, \; nT \leq \tau \leq (n+1)T \tag{3.2.6}$$

$$, \; n = \max\left\{k \,|\, k \leq \frac{\tau}{T}\right\}, \quad k = 0, 1, 2, \ldots$$

For the spectral density, we obtain

$$\phi_y(\omega) = \frac{\sin^2 \frac{\omega T}{2}}{(\frac{\omega T}{2})^2} \sum_{n=-\infty}^{\infty} \phi_x[\omega - (n + \tfrac{1}{2})\omega] \; , \quad \omega = \frac{2\pi}{T} \quad . \tag{3.2.7}$$

From equation (3.2.6), we obtain a simple graphical construction of $R_y(\tau)$ [see Fig. 3.3].

### 3.2.3 Frequency analysis

As analogues of equations (3.1.51) and (3.1.75), we obtain

$$R_y^*(s) = p^*(0,s)R_x(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_x(u)\,\psi_A'^{**}(-iu,s)\,du\,, \quad Re(s) > 0 \tag{3.2.8}$$

and

$$R_y^*(s) = p^*(0,s)R_x(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}R_x^*(p)\,\psi_A'^{**}(-p,s)\,du$$

$$p = r + iu, \quad s = v + i\omega \quad , \tag{3.2.9}$$

where

$$\psi_A'^{**}(-p,s) = \sum_{n=1}^{\infty}(-1)^n\,\psi_n'^{**}(-p,s)\,. \tag{3.2.10}$$

For the case of independent sampling intervals, we have

$$R_y^*(s) = \frac{R_x(0)}{s} - \frac{g_1^*(s)}{2\pi s}\int_{-\infty}^{\infty}\phi_x(u)\,\frac{1+f_1^*(-iu)}{1+f_1^*(s-iu)}\,du\,, \quad Re(s) > 0 \tag{3.2.11}$$

where

$$g_1^*(s) = \beta\,\frac{1-f_1^*(s)}{s} \tag{3.2.12}$$

or

$$R_y^*(s) = \frac{1-g_1^*(s)}{s}R_x(0) + \frac{g_1^*(s)}{2\pi s}\int_{-\infty}^{\infty}R_x^*(p)\,\frac{f_1^*(s-p)-f_1^*(-p)}{1+f_1^*(s-p)}\,du$$

$$p = r + iu, \quad s = v + i\omega \quad .$$

$$0 < r < \lambda\,, \quad r < v \tag{3.2.13}$$

Examples:

i) Poisson sampling

Evaluation of expression (3.2.13) by residues leads to

$$R_y^*(s) = \frac{R_x(0)}{\beta+s} + \frac{\beta^2}{(\beta+s)^2}\left[R_x^*(2\beta+s) - R_x^*(\beta)\right]. \tag{3.2.14}$$

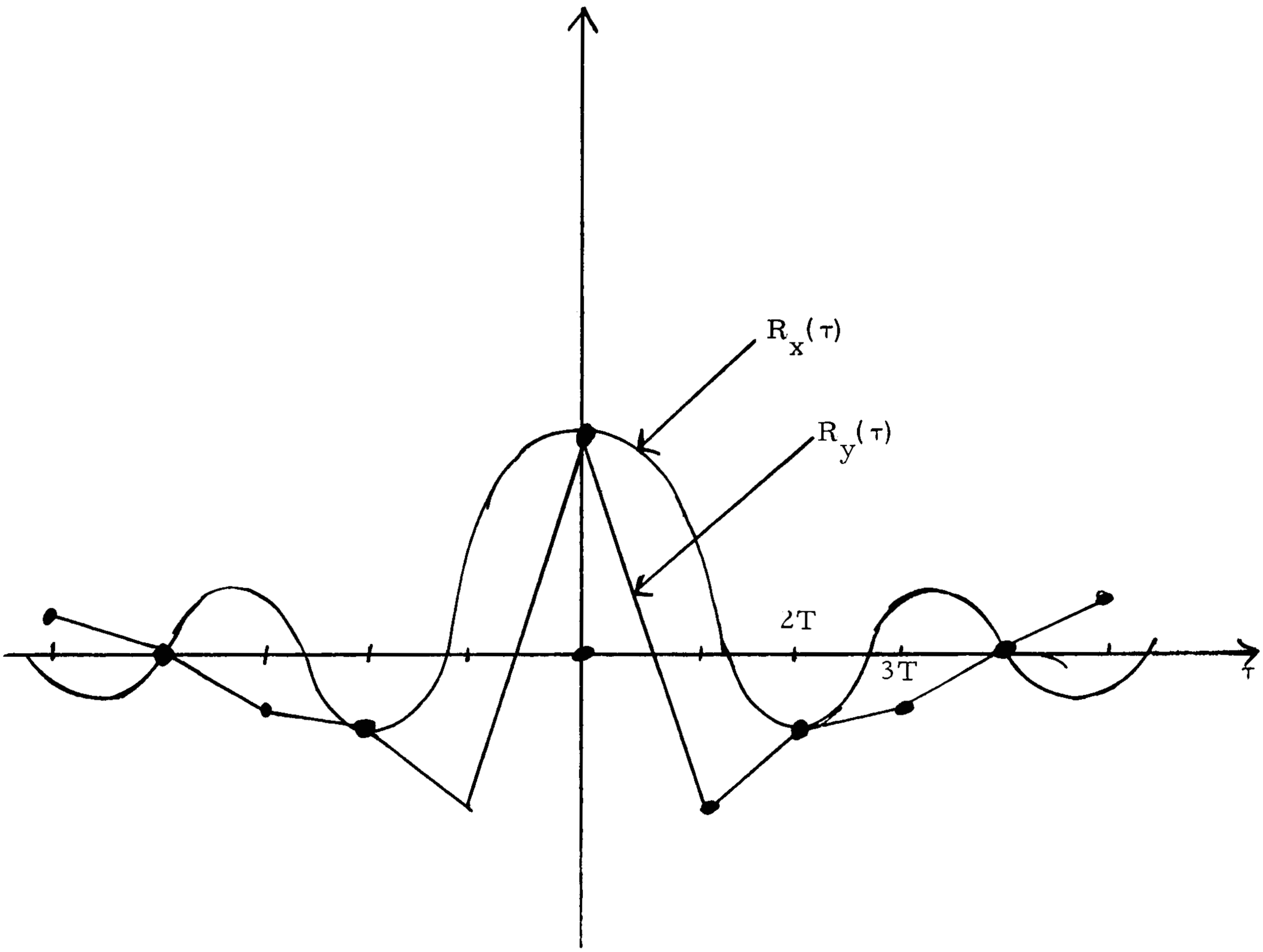


Figure 3. 3. Graphical construction of $R_y(\tau)$.

For illustration, if we take $x(t) = 1$, that is

$$R_x^*(s) = \frac{1}{s}$$

we obtain

$$R_y^*(s) = \frac{1}{2\beta + s}$$

or

$$\phi_y(\omega) = \frac{4\beta}{4\beta^2 + \omega^2}$$

as it should be, in view of the fact that $y(t)$ is the random telegraph wave.

ii) Binary process

If $x(t) = 1$, then $y(t)$ is a binary process taking on the values of $\pm 1$. With

$$R_x(\tau) = 1$$

or

$$\phi_x(u) = 2\pi\,\delta(u)$$

equation (3.2.11.) gives

$$R_y^*(s) = \frac{1}{s} - 2\beta\,\frac{1}{s^2}\,\frac{1 - f_1^*(s)}{1 + f_1^*(s)}\,, \quad Re(s) > 0 \,. \tag{3.2.15}$$

## 3.3 The random maneuver process

### 3.3.1 Definition

The random maneuver process $y(t)$ is the continuous parameter random process defined as

$$y(t) = \alpha_n\,, \quad t_n < t \leqslant t_{n+1} \tag{3.3.1}$$

where $\{\alpha_n\}$ denotes a stationary random process which is independent of the stationary point process $\{t_n\}$.

This process could describe the random maneuvers of a target in space, thus the designation.

### 3.3.2 Second-order statistics

Letting

$$\alpha = E[\alpha_n], \quad \rho(n) = E[\alpha_{n+m}\alpha_m] \quad , \tag{3.3.2}$$

we have

$$E[y(t)] = \alpha \tag{3.3.3}$$

and

$$E[y(t)^2] = \rho(0) \quad . \tag{3.3.4}$$

In view of

$$E[y(t+\tau)y(t)/A_n(t,\tau)] = \rho(n), \qquad \tau > 0$$

it follows that

$$R_y(\tau) = \sum_{n=0}^{\infty} \rho(n)\, p(n,\tau) \tag{3.3.5}$$

where the $p(n, \tau)$ are defined by equations (1.3.1) and (1.3.2). Also

$$R_y^*(s) = \sum_{n=0}^{\infty} \rho(n)\, p^*(n,s) \quad , \quad \mathrm{Re}(s) > 0 \tag{3.3.6}$$

Example:

Poisson sampling

From (1.3.97) and (1.4.6), we obtain

$$p^*(n,s) = \frac{1}{\beta+s}\left\{\frac{\beta}{\beta+s}\right\}^n$$

and therefore

$$R_y^*(s) = \frac{1}{\beta+s}\sum_{n=0}^{\infty} \rho(n)\left\{\frac{\beta}{\beta+s}\right\}^n \quad . \tag{3.3.7}$$

For the random telegraph wave, we choose

$$\rho(n) = (-1)^n$$

and we find

$$R_y^*(s) = \frac{1}{2\beta+s}$$

as it should be.

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# Appendix A

## Higher-order statistics for the impulse process $s(t) = \sum_{n=-\infty}^{\infty} \alpha_n \delta(t - t_n)$

Similarly to section 2.2., we can write

$$E[\, s(t)\, s(t+\tau_1)\, s(t+\tau_1+\tau_2)\,]$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \rho(n, m) \left\{ \lim_{d \to 0} \frac{P\left[ \begin{array}{l} (L_1 \le d) \cap (L_n \le \tau_1) \cap (\tau_1 < L_{n+1} \le \tau_1 + d) \cap \\ (L_{n+m} \le \tau_1 + \tau_2) \cap (\tau_1 + \tau_2 < L_{n+m+1} \le \tau_1 + \tau_2 + d) \end{array} \right]}{d^3} \right\},$$

for $\tau_1 > 0$, $\tau_2 > 0$ (A.1.)

where

$$\rho(n, m) = E[\alpha_1 \, \alpha_{n+1} \, \alpha_{n+m+1}] \qquad \text{(A.2.)}$$

and

$$L_o \equiv 0.$$

In the case of independent intervals, and after studying in detail the cases $|\tau_1| \le d$, $|\tau_2| \le d$, $|\tau_1 + \tau_2| \le d$, ... etc., we obtain, using some heuristics, that

$$E[\, s(t)\, s(t+\tau_1)\, s(t+\tau_1+\tau_2)\,] = \beta\rho(0,0)\,\delta(\tau_1)\,\delta(\tau_2) + \beta\delta(\tau_1) \sum_{m=1}^{\infty} \rho(0, m)\, f_m(\tau_2)$$

$$+ \beta\delta(\tau_2) \sum_{n=1}^{\infty} \rho(n, 0)\, f_n(\tau_1) + \beta \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \rho(n, m)\, f_n(\tau_1)\, f_m(\tau_2),$$

for $\tau_1 \ge 0_-$, $\tau_2 \ge 0_-$ (A.3.)

and where $f_n(\sigma)$ is defined by equation (2.1.30.).

In particular, if

$$\alpha_n = 1 \tag{A.4.}$$

the preceding expression can be written

$$E[s(t)s(t+\tau_1)s(t+\tau_1+\tau_2)) = \beta[\delta(\tau_1) + \sum_{n=1}^{\infty} f_n(\tau_1)][\delta(\tau_2) + \sum_{m=1}^{\infty} f_m(\tau_2)],$$

$$\text{for } \tau_1 \geq 0_-, \quad \tau_2 \geq 0_- \tag{A.5.}$$

The preceding results can be generalized to include higher-orders; for example, as in (A.5.), we also have

$$E[s(t)s(t+\tau_1)\ldots s(t+\tau_1+\ldots+\tau_{k-1})] = \beta \prod_{j=1}^{k-1} [\delta(\tau_j) + \sum_{n=1}^{\infty} f_n(\tau_j)],$$

$$\text{for } \tau_1 \geq 0_-, \ldots \tau_{k-1} \geq 0_- \tag{A.6.}$$

<u>Illustration:</u> <u>Poisson point process</u>

In this case, equation (A.6.) gives the very simple expression

$$E[s(t)s(t+\tau_1)\ldots s(t+\tau_1+\ldots+\tau_{k-1})] = \beta \prod_{j=1}^{k-1} [\delta(\tau_j) + \beta]. \tag{A.7.}$$

This result is useful as a means of obtaining higher order statistics for the secondary process

$$y(t) = \sum_{n=-\infty}^{\infty} \eta((t-t_n)\,. \tag{A.8.}$$

From

$$y(t) = \int_{-\infty}^{\infty} \eta(t-\tau)\, s(\tau)\, d\tau$$

and using equation (A.7.), we obtain as an example

$$E[y(t)^3] = \beta \int_{-\infty}^{\infty} \eta(\sigma)^3\, d\sigma + 2\beta^2 \left\{ \int_{-\infty}^{\infty} \eta(\sigma)\, d\sigma \right\} \left\{ \int_{-\infty}^{\infty} \eta(\sigma)^2\, d\sigma \right\} + \beta^3 \left\{ \int_{-\infty}^{\infty} \eta(\sigma)\, d\sigma \right\}^3 \tag{A.9.}$$

## Appendix ~~B~~ A

An intuitive intrepretation for the distribution functions $F_n(x)$.

Let us consider a stationary point process such that

$$\lim_{h \to 0} \frac{G_1(h)}{h} = \beta \qquad (A.1.)$$

$$\lim_{h \to 0} \frac{G_2(h)}{h} = 0 \ . \qquad (A.2.)$$

As in (1.2.14.), (1.2.15), we can write

$$P\left[x < L_n(t) \leq x + dx\right] = \sum_{k=0}^{n-1} P\left[E_{n-k}(t+x,\ dx)\right] P\left[\frac{Ak(t,\ x)}{E_{n-k}(t+x,\ dx)}\right] ,$$

$dx > 0, \ x > 0$

and in view of the preceding assumptions, we obtain

$$g_n(x) = \beta \lim_{dx \to 0} P\left[\frac{A_{n-1}(t,\ x)}{E_1(t+x,\ dx)}\right] \ . \qquad (A.3.)$$

Comparison with equations (1.3.36.), (1.3.37.) shows that

$$1 - F_1(x) = \lim_{dx \to 0} P\left[\frac{A_o(t,\ x)}{E_1(t+x,\ dx)}\right] \qquad (A.4.)$$

and

$$F_{n-1}(x) - F_n(x) = \lim_{dx \to 0} P\left[\frac{A_{n-1}(t,\ x)}{E_1(t+x,\ dx)}\right] \ . \qquad (A.5.)$$

In order to interpret the right-hand sides of (B.4.), (B.5.), we shall introduce the notion of a conditional description of a stationary point process: let us consider an enumerable sequence of events which constitute a stationary point process; suppose we are given that one or more events occur at the instant $t = t_o$; under these conditions [and as in section 1.1.1.], we define the random intervals

$$X_n(t_o) = t_n - t_{n-1} \geq 0, \quad n=0, \pm 1, \pm 2, \qquad \text{(B.6.)}$$

and say that the random process $\{X_n(t_o)\}$ provides a conditional description of the stationary point process.

Discussion.

We would like to point out that the preceding concept [conditional description] in ambiguous and arbitrary: indeed, what is actually

$$P[X_o(t_o) > \tau] \ ?$$

Using definitions (1.2.1.), (1.2.11.), we can write

$$\{X_o(t_o) > \tau\} = \left\{\frac{A_o(t_o - \tau, \tau)}{E_1(t_o, 0)}\right\}$$

and since

$$P[E_1(t_o, 0)] = 0 \quad ,$$

we have therefore

$$P[X_o(t_o) > \tau] = \frac{0}{0} \ . \qquad \text{(B.7.)}$$

Yet, one can give a sensible meaning to (B.7.) i) by limiting procedures:* one could define the value of (B.7.) as

$$P[X_o(t_o) > \tau] = \lim_{h \to 0} P\left[\frac{A_o(t_o - \tau, \tau)}{E_1(t_o, h)}\right] \qquad \text{(B.8.)}$$

or

$$P[X_o(t_o) > \tau] = \lim_{h \to 0} P\left[\frac{A_o(t_o - \tau h, \tau)}{E_1(t_o - h, h)}\right] \qquad \text{(B.9.)}$$

* This is the usual procedure.

and so on. Unfortunately, all these limits will be, in general, different: which one, if any, is the right one ?

ii) by sampling procedures:

Suppose we are given a sample of the stationary point process. Imagine yourself measuring the time intervals between consecutive occurrences, thus obtaining an empirical [or "time-averaging"] distribution function $P[X_o \leq x]$, where $X_o$ denotes the time interval between any two consecutive occurrences. Then the value of (B.7.) could be defined as

$$P[X_o(t_o) > \tau] = P[X_o > \tau] . \qquad \text{(B.10.)}$$

How do we reconcile these different views ? This seems very difficult and perhaps impossible.

However, there are some known stationary point processes [for instance, the periodic point process where $X_o = T$] for which the different values obtained from (B.8.), (B.9.), (B.10.) all agree, thus giving a somewhat legitimate meaning to $X_o(t_o)$. In such a case, it would follow from (B.10.), B 8.), (B.4.) that

$$F_1(x) = P[X_o \leq x] \qquad \text{(B.11.)}$$

thus providing a possible interpretation for $F_1(x)$. Moreover, notice that in view of (1.3.61.), we have

$$\beta = \frac{1}{E[x_o]} . \qquad \text{(B.12.)}$$

The preceding discussion can be repeated for all the $X_n(t_o)$ and similarly it may be seen that

$$F_n(x) = P[X_o + X_1 + \ldots + X_{n-1} \leq x] \qquad \text{(B.13.)}$$

where $X_o + X_1 + \ldots + X_{n-1}$ denotes the time interval between any occurrence and the $n^{th}$ following one.

# REFERENCES


A. 1. Adomian, G. "Linear stochastic operators" Doctoral thesis, Dept. of physics, 1963, U. C. L. A., Los Angeles, California.

B. 1. Balakrishnan, A. V. "A note of the sampling principle for continuous signals" PGIT, no. 3, 1957.

B. 2. Balakrishnan, A. V. "On the problem of time-jitter in sampling" PGIT, no. 3, 1962.

B. 3. Bartlett, M. S. "The spectral analysis of point processes" Royal Statistical Society, series B, vol. 25, no 2, 1963.

B. 4. Bergen, A. R. "The synthesis of optimum random sampling systems" Doctoral thesis, 1958, Electrical Engineering Department, Columbia University, New York.

B. 5. Bergen, A. R. "On the statistical design of linear random sampling systems" IFAC, vol. I, 1960, Moscow.

B. 6. Beutler, F. J. "Sampling theorems and bases in Hilbert spaces" Information and Control, vol. 4, 1961.

B. 7. Beutler, F. J. "A generalization of Wiener optimum filtering and prediction" Doctoral thesis, California Institute of Technology, 1957.

B. 8. Blanc-Lapierre, A. "Sur certaines fonctions aleatoires stationaires. Application à l'etude des fluctuations dues à la structure electronique de l'electricité" Masson et C$^{ie}$, 1945.

B. 9. Brown, W. "Sampling with random jitter" SIAM, vol. 11, 1963.

C. 1. Cox, D. R. and Smith, W. I. "On the superposition of renewal processes" Biometrika, no. 1, 1954.

D. 1. Davenport, W. and Root, W. L. "An introduction to the theory of random signals and noise" McGraw-Hill, 1958.

D. 2. Ditkin, V. A. and Prudnikov, A. P. "Operational calculus in two variables and its applications" Pergamon press, the MacMillan Company, New York.

D. 3. Doetsch, G. "Guide to the application of the Laplace transforms" Van Nostrand.

D. 4. Doob, J. "Stochastic processes" Wiley.

E. 1. Erdelyi, A. "From Delta functions to distributions" Modern mathematics for the engineer, second series. McGraw-Hill, 1961.

E. 2. Erdelyi, A. "Operational calculus and generalized functions" Holt, Rinehart and Winston, New York, 1962.

F. 1. Feller, W. "An introduction to probability theory and its applications" Wiley, 1961.

G. 1. Gel'fand, I. M. and Shilov, G. E. "Les distributions" Dunod, Paris, 1962.

G. 2. Goldberg, R. "Fourier transforms" Cambridge at the University Press, 1961.

G. 3. Graves, L. "The theory of functions of real variables" McGraw-Hill, 1946.

H. 1. Hardy, G. H., Littlewood, J. E. and Polya, G. "Inequalities" Cambridge at the University Press, 1934.

H. 2. Hardy, G. H. "Divergent series" Oxford, Clarendon Press, 1949.

H. 3. Hobson, E. W. "The theory of functions of a real variable and the theory of Fourier's series" Vol. I and II. Cambridge at the University Press, 1927.

K. 1. Kalman, R. E. "Analysis and synthesis of linear systems operating on randomly sampled-data" Doctoral thesis, Electrical Engineering Department Columbia University, New York, 1957.

K. 2. Knopp, K. "Theory and application of infinite series" Blackie and Son, 1928.

K. 3. Korevaar, J. "Distributions defined by fundamental sequences" Nederl. Akad. Wetensch. Proc. Ser. A, vol. 58, 1955.

L. 1. Laning, J. H. and Battin, R. H. "Random processes in automatic control" McGraw-Hill, New York, 1956.

L. 2. Lavoine, J. "Calcul symbolique: Distributions et Pseudo-fonctions" C. N. R. S., 1959.

L. 3. Leneman, O. A. "Level-crossings by sample functions of a stochastic process" Minor thesis, Instrumentation Engineering Department, University of Michigan, Ann Arbor, 1963.

L. 4. Lighthill, M. J. "Introduction to Fourier Analysis and generalized functions" Cambridge at the University Press. 1959.

L. 5. Loeve, M. "Probability theory" Van Nostrand, 1955.

L. 6. Lukaes, E. "Characteristic functions" Hafner Publishing Company, New York, 1960.

M. 1. McFadden, J. A. "On the lengths of intervals in a stationary point process" Royal Statistical Society, series B, vol. 24, no. 2, 1962.

M. 2. McFadden, J. A. and Weissblum, W. "Higher order properties of a stationary point process" Royal Statistical Society, vol. 25, series B, no. 2, 1963.

N. 1. Natanson, I. P. "Theory of functions of a real variable" vol. I and II Frederick Ungar Publishing Co., New York.

N. 2. Nobel, B. "Methods based on the Wiener-Hopf technique for the solution of partial differential equations" Pergamon Press 1958.

P. 1. Palermo, C. "Theory of stochastic delay" Doctoral thesis, Electrical Engineering Department, University of Michigan, Ann Arbor, 1962.

P. 2. Parzen, E. "Stochastic processes" Holden-Day, 1962.

R. 1. Ragazini, J. and Franklin G. "Sampled-data control systems" McGraw-Hill, 1958.

R. 2. Riesz, F. and S. Z. Nagy, B. "Functional Analysis" Ungar publishing Co.

S. 1. Schwartz, L. "Theorie des distributions" 2 vols. Hermann et C$^{ie}$, Paris, 1950-1951.

S. 2. Shapiro, H. A. and Silverman, R. A. "Alias-free sampling of random noise" SIAM, vol. 8, no. 2, 1960.

S. 3. Slepian, D. "First passage time for a particular gaussian process" Annals of Math. Stat. vol. 32, 1962.

S. 4. Smith, W. L. "Renewal theory and its ramifications" Royal Statistical Society, series B, vol. 20, 1958.

T. 1. Takacs, L. "Stochastic processes: Problems and solutions" Methuen, London, 1960.

T. 2. Takacs, L. "Introduction to the theory of Queues" Oxford University Press, New York, 1962.

W. 1. Widder, D. "The Laplace transform" Princeton University Press, 1946.

W.2. Wiener, N. "The Fourier integral and certain of its applications" Dover Publications.

W.3. Wiener, N. "Extrapolation, interpolation and smoothing of stationary time series" Wiley and sons, New York, 1949.

W.4. Wilder, R. "Introduction to the foundations of mathematics" Wiley and sons.

W.5. Wold, H. "Sur les processus stationnaires ponctuels" Colloques internationaux du CNRS: Le calcul des probabilités et ses applications, Lyon, 1948.

Y. 1. Yaglom, A. M. "An introduction to the theory of stationary random processes" Prentice Hall, Inc., 1962.

Z. 1. Zigmund, A. "Trigonometric series" Vols. I, II, Cambridge at the University Press, 1959.

## Glossary of Principal Symbols

The symbols listed below are those frequently used throughout this dissertation. Symbols defined and used incidentally in the derivation of formulae are not included.

$A_n(t, \tau) = \left\{\begin{array}{l}\text{n occurrences in} \\ \text{the interval } (t, t+\tau]\end{array}\right\}, \quad \tau > 0$

$a_n$ = Amplitude error in sampling

$C(i\omega)$ = $E[e^{-i\omega e_n}]$

$E[\ ]$ = Stands for statistical expectation

$E_n(t,\tau) = \left\{\begin{array}{l}\text{at least n occurrences} \\ \text{in the interval } (t, t+\tau]\end{array}\right\}, \quad \tau > 0$

$e_n$ = Time-jitter error before sampling

$F(\sigma, \tau)$ = Defined by (3.1.16)

$F_n(x)$ = Defined by (1.3.36), (1.3.37)

$f_n(x) \doteq \dfrac{dF_n(x)}{dx}$ a. e.

$f_n^*(s) = \displaystyle\int_{0^-}^{\infty} e^{-sx}\, dF_n(x)$

[In the case of independent intervals, we have

$f_n^*(s) = \{f_1^*(s)\}^n = \{E[e^{-sx_n}]\}^n$ ]

$G_n(\tau)$ = $P[L_n \le \tau]$

$g_n(x) = \dfrac{d\,G_n(x)}{dx}$ a. e.

$g_n^*(s) = \displaystyle\int_{0^-}^{\infty} e^{-sx}\, d\,G_n(x)$

$= E[e^{-sL_n}]$

| | | |
|---|---|---|
| $H(\omega)$ | = | Transfer function for the optimum linear interpolator |
| $h(t)$ | = | Interpolation function |
| $L_n$ | = | Passage time for the $n^{th}$ forward occurrence |
| $L_{-n}$ | = | Time elapsed since the $n^{th}$ backward occurrence |
| $N(t, \tau)$ | = | Random number of occurrences in the interval $(t, t+\tau]$ |
| $P[\;]$ | = | Probability measure |
| $p(n, \tau)$ | = | $P[N(t, \tau) = n]$ |
| $q$ | = | The probability of skipping an occurrence |
| $R_s(\tau)$ | = | Generalized correlation function for $s(t)$ |
| $R_x(\tau)$ | = | Correlation function for $x(t)$ |
| $R_x^*(s)$ | = | $\int_0^\infty e^{-s\tau} R_x(\tau)\, d\tau$ |
| $s(t)$ | = | The impulse process, i. e., $s(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, \delta(t - t_n)$ |
| $T$ | = | Time interval for the periodical point process |
| $x_n$ | = | $L_{n+1} - L_n$ |
| $\alpha_n$ | = | Modulating factors in the impulse process |
| $\alpha$ | = | $E[\alpha_n]$ |
| $\beta$ | = | Average number of occurrences per unit time |
| $\epsilon_n$ | = | Time-jitter error after sampling |
| $\phi_s(\omega)$ | = | Generalized spectral density for the impulse process |
| $\phi_x(\omega)$ | = | Spectral density for $x(t)$ [possibly generalized] |
| $\psi'(\sigma, \tau)$ | = | $\sum_{n=1}^{\infty} \psi_n(\sigma, \tau) = \dfrac{d\psi(\sigma, \tau)}{d\sigma}$ a. e. |

$\psi_n'(\sigma, \tau)$ = Defined by (3.1.20)

$$\psi_n'^{**}(-p, s) = \int_0^\infty e^{-s\tau} \int_0^\infty e^{p\sigma} \psi_n'(\sigma, \tau)$$

$$\psi'^{**}(-p, s) = \sum_{n=1}^{\infty} \psi_n'^{**}(-p, s)$$

$$\psi_A'^{**}(-p, s) = \sum_{n=1}^{\infty} (-1)^n \psi'^{**}(-p, s)$$

$\gamma(i\omega) = E[e^{-i\omega\epsilon_n}]$

$\psi(\sigma, \tau)$ = Defined by (3.1.12)

$\rho(n) = E[\alpha_{m+n}\alpha_m]$

$\sigma^2 = E[a_n^2]$

$\omega_0 = \dfrac{2\pi}{T}$

Errata Sheet.

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8 March 1965

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Miss Winnie M. Morgan
Technical Reports Office
National Aeronautics and Space Administration
Washington, D. C. 20546

N65-18048

Re: SC-NsG-2/23-05-001

Dear Miss Morgan:

I have received your letter of March 2, 1965 concerning the publication of my Technical Report, "Stationary Point Processes and Their Application to Random Sampling of Stochastic Processes." There are two minor changes I would like to make.

1) I wish to eliminate Appendix A, both in the Table of Contents and also on pages 109-110. Appendix B, pages 111-113 will therefore become Appendix A, thus (B.1) will become (A.1), etc....

2) Also, Eq. (2.4.49) page 72 should be

$$h(t) = \begin{cases} \dfrac{e^{-a|t|} - e^{a|t|}\, e^{-2aT}}{1 - e^{-2aT}} , & 0 < |t| \leq T \\ 0 , & \text{otherwise} \end{cases}$$

I wish to thank the National Aeronautics and Space Administration for publishing my report and I shall be glad to be of further help if it is needed.

Sincerely yours,

O.A.Z. Leneman

Dr. O. A. Z. Leneman

OAZL:smm